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Maximal arcs in projective planes of order 16 and related designs

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Abstract: The resolutions and maximal sets of compatible resolutions of all 2-(120, 8, 1) designs arising from maximal (120, 8)-arcs, and the 2-(52, 4, 1) designs arising from previously known maximal (52, 4)-arcs, as well as some newly discovered maximal (52, 4)-arcs in the known projective planes of order 16, are computed. It is shown that each 2-(120, 8, 1) design associated with a maximal (120, 8)-arc is embeddable in a unique way in a projective plane of order 16. This result suggests a possible strengthening of the Bose–Shrikhande theorem about the embeddability of the complement of a hyperoval in a projective plane of even order. The computations of the maximal sets of compatible resolutions of the 2-(52, 4, 1) designs associated with maximal (52, 4)-arcs show that five of the known projective planes of order 16 contain maximal arcs whose associated designs are embeddable in two nonisomorphic planes of order 16.

Keywords: Maximal arc, projective plane, resolvable design.

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1 Introduction

A 2-(v, k, λ) design (or shortly, a 2-design) is a pair $D = (X, B)$ of a set X of v *points* and a collection B of subsets of X of size k called *blocks*, such that every two points appear together in exactly λ blocks; see [4; 10]. Every point of a 2-(v, k, λ) design is contained in $r = \lambda(v-1)/(k-1)$ blocks, and the total number of blocks is $b = v(v-1)\lambda/k(k-1)$.

The incidence matrix of a design D is a (0, 1)-matrix $A = (a_{ij})$ with rows labeled by the blocks, columns labeled by the points, where $a_{i,j} = 1$ if the ith block contains the jth point, and $a_{i,j} = 0$ otherwise. If p is a prime, the p-rank of a design D is the rank of its incidence matrix over a finite field of characteristic p.

Two designs are *isomorphic* if there is a bijection between their point sets that maps every block of the first design to a block of the second design. An *automorphism* of a design is any isomorphism of the design to itself. All automorphisms of D form the automorphism group $\mathrm{Aut}(D)$ of D.

The *dual* design $D^\perp$ of a design D has as points the blocks of D, and as blocks the points of D. A 2-(v, k, λ) design is *symmetric* if $b = v$, or equivalently, $r = k$. The dual design $D^\perp$ of a symmetric 2-(v, k, λ) design D is a symmetric design with the same parameters as D. A symmetric design D is *self-dual* if D and $D^\perp$ are isomorphic.

A design with $\lambda = 1$ is called a *Steiner* design. An affine plane of order n, where $n \geq 2$, is a Steiner 2-$(n^2, n, 1)$ design. A projective plane of order n is a symmetric Steiner 2-$(n^2+n+1, n+1, 1)$ design with $n \geq 2$. The classical (or Desarguesian) plane $\mathrm{PG}(2, p^t)$ of order $n = p^t$, where p is prime and $t \geq 1$, has as points the 1-dimensional subspaces of the 3-dimensional vector space V_3 over the finite field of order p^t, and as blocks (or *lines*), the 2-dimensional subspaces of V_3.

Let $D = (X, B)$ be a Steiner 2-$(v, k, 1)$ design with point set X, collection of blocks B, and let v be a multiple of k, say $v = nk$. Since every point of X is contained in $r = (v-1)/(k-1) = (nk-1)/(k-1)$ blocks, it follows that $k-1$ divides $n-1$. Thus, $n-1 = s(k-1)$ for some integer $s \geq 1$, and $v = nk = (sk-s+1)k$. A *parallel class*

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P is a set of $v/k = n$ pairwise disjoint blocks, and a *resolution* of D is a partition of the collection of blocks B into $r = (v-1)/(k-1) = sk+1$ parallel classes. A design is *resolvable* if it admits a resolution.

Any 2-$((sk-s+1)k, k, 1)$ design with $s = 1$ is an affine plane of order k, and admits exactly one resolution. If $s > 1$, a resolvable 2-$((sk-s+1)k, k, 1)$ design may admit more than one resolution. Following [31], we call two resolutions R_1, R_2,

$$R_1 = P_1^{(1)} \cup P_2^{(1)} \cup \cdots \cup P_r^{(1)}, \quad R_2 = P_1^{(2)} \cup P_2^{(2)} \cup \cdots \cup P_r^{(2)} \tag{1}$$

compatible if they share one parallel class, $P_i^{(1)} = P_j^{(2)}$, and $|P_{i'}^{(1)} \cap P_{j'}^{(2)}| \le 1$ for $(i', j') \ne (i, j)$. More generally, a set of m resolutions $R_1, \dots, R_m$ is compatible if every two of these resolutions are compatible.

Let P be a projective plane of order $q = sk$. A *maximal* $((sk-s+1)k, k)$*-arc*, or a maximal arc of degree k, is a set A of $(sk-s+1)k$ points of P such that every line of P is either disjoint from A or meets A in exactly k points; cf. [22]. A maximal arc of degree k is *nontrivial* if $2 \le k < q$. A *hyperoval* is a maximal arc of degree 2. The collection of lines of P which have no points in common with A determines a maximal $((sk-k+1)s, s)$-arc $A^\perp$ (called a *dual arc*) in the dual plane $P^\perp$.

Maximal arcs with $1 < k < q$ do not exist in any Desarguesian plane of odd order q by [3], and are known to exist in every Desarguesian plane of order $q = 2^t$, see [12; 11; 19; 20; 24], as well as in some non-Desarguesian planes of even order, see [18; 17; 15; 16; 21; 28; 29; 30].

If $k > 1$, the nonempty intersections of a maximal $((sk-s+1)k, k)$-arc A with the lines of a projective plane P of order $q = sk$ are the blocks of a resolvable 2-$((sk-s+1)k, k, 1)$ design D. Similarly, if $s > 1$, the corresponding $((sk-k+1)s, s)$-arc $A^\perp$ in the dual plane is the point set of a resolvable 2-$((sk-k+1)s, s, 1)$ design $D^\perp$. We refer to D (respectively $D^\perp$) as a design embeddable in P (respectively $P^\perp$) as a maximal arc. The points of $D^\perp$ determine a set of $(sk-k+1)s$ mutually compatible resolutions of D. Respectively, the points of D determine a set of $(sk-s+1)k$ mutually compatible resolutions of $D^\perp$.

Two maximal arcs A', A'' in a projective plane P are *equivalent* if there is an automorphism of P that maps A' to A''. We note that the designs associated with equivalent arcs are necessarily isomorphic, while the converse is not true in general.

The following theorem, proved recently by one of the authors [31], gives an upper bound on the number of pairwise compatible resolutions of a 2-$((sk-s+1)k, k, 1)$ design, and characterizes the designs for which this upper bound is achieved.

Theorem 1.1 ([31]). *Let $S = \{R_1, \dots, R_m\}$ be a set of m mutually compatible resolutions of a 2-$((sk-s+1)k, k, 1)$ design $D = (X, B)$. Then $m \le (sk-k+1)s$. The equality $m = (sk-k+1)s$ holds if and only if there exists a projective plane P of order sk such that D is embeddable in P as a maximal $\{(sk-s+1)k, k\}$-arc.*

The possible values for the degree k of a nontrivial maximal arc in a projective plane of order 16 are 2, 4, and 8, and the parameters of the Steiner designs associated with such arcs are 2-$(18, 2, 1)$, 2-$(52, 4, 1)$, and 2-$(120, 8, 1)$, respectively. Clearly, a 2-$(18, 2, 1)$ associated with a hyperoval H is the trivial design having as blocks the unordered pairs of points of H, or equivalently, the edges of the complete graph on 18 vertices being the points of H.

This paper summarizes the computation of all parallel classes, resolutions, and compatible sets of resolutions of maximum size of the 2-$(52, 4, 1)$ designs associated with maximal $(52, 4)$-arcs, and the 2-$(120, 8, 1)$ associated with maximal $(120, 8)$-arcs in the known projective planes of order 16. The main results are:

Theorem 1.2.
(i) *Every 2-$(120, 8, 1)$ design associated with a maximal $(120, 8)$-arc in a known projective plane P of order 16 admits exactly one compatible set of resolutions of maximal size, meeting the bound of Theorem 1.1, and consequently it is uniquely embeddable in P.*
(ii) *Five of the known projective planes of order 16 (the Lorimer–Rahilly plane LMRH, its dual plane $LMRH^\perp$, the Johnson plane JOHN, the plane BBH1 obtained by Bose–Barlotti derivation [5] of the Hall plane, and the Johnson–Walker plane JOWK), contain maximal $(52, 4)$-arcs whose associated 2-$(52, 4, 1)$ designs admit two different sets of 52 compatible resolutions, and are embeddable in two different planes.*

(iii) *Each of the following pairs* $\{P_1, P_2\}$ *of projective planes of order* 16 *contain maximal* $(52, 4)$*-arcs such that the associated 2-*$(52, 4, 1)$ *designs are isomorphic:*
- *the Johnson plane and the Johnson–Walker plane;*
- *the Johnson plane and the plane BBH1;*
- *the Lorimer–Rahilly plane LMRH and its dual plane.*

Part (i) of Theorem 1.2 gives rise to the following

Conjecture 1.3. *Every 2-*$(\binom{q}{2}, \frac{q}{2}, 1)$ *design D associated with a maximal* $(\binom{q}{2}, \frac{q}{2})$*-arc A in a projective plane P of even order q is uniquely embeddable in P.*

We note that a $(\binom{q}{2}, \frac{q}{2})$-arc in a plane of even order q is a dual arc of a hyperoval in $P^{\perp}$ (for this reason, the associated 2-$(\binom{q}{2}, \frac{q}{2}, 1)$ design is called an *oval* design in [1, 8.4]). Since each of the planes of order 2, 4, and 8 contains exactly one equivalence class of hyperovals, the answer of the above question is in the affirmative (trivially) for $q = 2, 4$, and 8.

Conjecture 1.3, if true, would be a significant strengthening of a theorem due to Bose and Shrikhande [6], stating that the complement of a hyperoval, that is, a maximal $(q+2, 2)$-arc, in any projective plane P of even order q is uniquely embeddable in P, because the conjecture assumption is weaker than the assumption of Bose–Shrikhande's theorem: for, the complement of a hyperoval contains q^2-1 points, while the complement of a $(\binom{q}{2}, \frac{q}{2})$-arc contains $(q^2+q+2)/2$ points, that is, $(q^2-q-4)/2$ points less than the complement of a hyperoval.

Part (ii) of Theorem 1.2 provides new connections between the known projective planes of order 16 (compare with the diagram of known connections given by Moorhouse in [26]). Part (iii) of Theorem 1.2 implies that the points and lines of the corresponding pair of planes $\{P_1, P_2\}$ can be reordered in such a way that the resulting planes $\{P_1', P_2''\}$ share a maximal $(52, 4)$-arc A, and the incidence matrices of the designs D', D'' associated with A are identical. Lists with the lines of planes isomorphic to the pairs of planes from Theorem 1.2, Part (iii), which share a maximal $(52, 4)$-arc, are available at www.math.mtu.edu/~tonchev/arcs.htm.

2 Maximal $(120, 8)$-arcs and related 2-$(120, 8, 1)$ designs

There are 22 nonisomorphic projective planes of order 16 that are known currently. Four planes, PG$(2, 16)$, SEMI2, SEMI4, and BBH1 (in the notation of [28]) are self-dual, and there are nine planes which are not self-dual: HALL, LMRH, JOWK, DSFP, DEMP, BBH2, JOHN, BBS4, and MATH; see [28]. Lists with the collections of lines of these planes are available at Eric Moorhouse's web page [26].

In [28], Penttila, Royle, and Simpson enumerated and classified up to equivalence all hyperovals in the known planes of order 16. A hyperoval A in a plane P of order 16 is a maximal $(18, 2)$-arc, and its dual arc $A^{\perp}$ is a maximal $(120, 8)$-arc in the dual plane $P^{\perp}$. Since two maximal arcs A' and A'' in a plane P are equivalent if and only if their dual arcs $(A')^{\perp}$, $(A'')^{\perp}$ are equivalent, the results from [28] imply the classification of all maximal $(120, 8)$-arcs in the known projective planes of order 16, up to equivalence.

We used the data about the inequivalent hyperovals graciously provided to the authors by Gordon F. Royle, to compute the corresponding dual $(120, 8)$-arcs and the related 2-$(120, 8, 1)$ designs. The 93 inequivalent hyperovals give rise to 93 inequivalent $(120, 8)$-arcs. For each 2-$(120, 8, 1)$ design D associated with an arc in the dual plane of the plane containing the corresponding hyperoval, we computed all parallel classes of D, all resolutions of D, and all compatible sets of maximal size 18. The parallel classes were found as 13-cliques in a graph Γ having as vertices the blocks of D, where two blocks are adjacent in Γ if they are disjoint. The resolutions were computed as 17-cliques in a graph Δ having as vertices the parallel classes of D, where two parallel classes are adjacent in Δ if they do not share any block. Finally, we computed the compatible sets of maximal size 18 as 18-cliques in a graph E having as vertices the resolutions of D, where adjacency is defined according to the definition of compatible resolutions given in the preceding section. For these computations, we wrote algorithms using Magma [7] and Cliquer [27]. The results of these computations are summarized in [32, Table 2.4].

Remark 2.1. The number of parallel classes ranges from 153 to 221, cf. [32, Table 2.4], while the number of resolutions is 18 in all but one notable exception, namely the 2-(120, 8, 1) oval design corresponding to the dual (120, 8)-arc of the regular hyperoval in PG(2, 16), with automorphism group of order 16, 320. The number of resolutions of this particular design is 137. However, the set of 137 resolutions contains only one set of 18 pairwise compatible resolutions, thus this design, as well as all remaining oval designs, are embeddable in a unique way in a projective plane of order 16.

The unique embeddability of the 2-(120, 8, 1) oval designs implies that the 273×273 incidence matrix of the related projective plane can be recovered uniquely (up to a permutation of the rows or columns), from the 120×255 incidence matrix of an oval design, which is a much smaller matrix than the 255×273 incidence matrix of the complement of a hyperoval (see Conjecture 1.3 in the previous section).

Remark 2.2. The 2-ranks of the 2-(120, 8, 1) designs associated with maximal (120, 8)-arcs range from 65 to 94, and the minimum 65 is achieved only by the two designs in the Desarguesian plane PG(2, 16) corresponding to the regular hyperoval with a group of order 16, 320, and the Lunelli–Sce hyperoval (see [23], also known as the Hall hyperoval [14]), with a group of order 144. This supports a conjecture by Carpenter [9], stating that the 2-rank of any design associated with a hyperoval in $PG(2, 2^t)$ is $3^t - 2^t$, as well as the following stronger conjecture formulated in [31], which generalizes a conjecture by Brouwer [8].

Conjecture 2.3 ([31]). *If D is a 2-$(2^{2t-1} - 2^{t-1}, 2^{t-1}, 1)$ design ($t \geq 2$), with an incidence matrix A, then* $\mathrm{rank}_2(A) \geq 3^t - 2^t$, *and the equality* $\mathrm{rank}_2(A) = 3^t - 2^t$ *holds if and only if D is embeddable as a maximal $(2^{2t-1} - 2^{t-1}, 2^{t-1})$-arc in* $PG(2^t, 2)$.

Conjecture 2.3 is trivially true for $t = 2$, and its validity for $t = 3$ follows from the results of [25].

3 Maximal (52, 4)-arcs and related 2-(52, 4, 1) designs

A list of all previously known maximal (52, 4)-racs in the known non-Desarguesian planes of order 16 is given in [21]. The following eight new arcs were found recently by Mustafa Gezek [13]:

MATH.2* = {260, 261, 259, 266, 20, 104, 74, 211, 170, 109, 214, 244, 32, 49, 129, 158, 117, 143, 63, 203, 40, 150, 251, 78, 69, 240, 198, 120, 17, 54, 241, 51, 175, 141, 136, 79, 46, 218, 212, 144, 197, 100, 106, 123, 21, 157, 115, 171, 160, 35, 205, 30}

MATH.3* = {53, 139, 113, 207, 256, 257, 265, 258, 13, 172, 179, 18, 42, 208, 148, 110, 73, 247, 86, 232, 54, 91, 229, 121, 136, 241, 79, 199, 0, 109, 170, 190, 34, 211, 156, 20, 9, 253, 192, 155, 183, 209, 52, 67, 111, 82, 126, 236, 37, 24, 166, 138}

MATH.4* = {256, 257, 258, 265, 15, 220, 118, 165, 234, 27, 45, 57, 254, 98, 147, 200, 64, 84, 177, 135, 8, 76, 242, 202, 237, 182, 213, 169, 83, 107, 23, 48, 47, 116, 145, 142, 14, 156, 85, 51, 235, 34, 250, 121, 199, 104, 176, 31, 214, 68, 161, 141}

MATH.5* = {263, 271, 270, 268, 0, 229, 190, 216, 91, 61, 102, 131, 22, 37, 206, 24, 243, 43, 192, 253, 7, 138, 58, 183, 226, 82, 111, 223, 5, 187, 224, 134, 94, 56, 99, 221, 10, 198, 40, 251, 239, 90, 55, 35, 103, 205, 210, 30, 21, 191, 240, 130}

MATH.6* = {256, 258, 257, 265, 5, 17, 224, 244, 187, 74, 175, 94, 6, 18, 25, 184, 179, 172, 13, 167, 42, 62, 208, 148, 122, 128, 196, 110, 39, 51, 194, 153, 104, 141, 214, 124, 43, 63, 243, 92, 149, 89, 72, 129, 231, 223, 77, 97, 226, 117, 203, 246}

MATH.7* = {260, 261, 259, 266, 0, 235, 201, 156, 190, 85, 34, 119, 22, 253, 77, 243, 168, 24, 166, 67, 7, 236, 185, 92, 9, 82, 226, 183, 11, 224, 153, 204, 181, 94, 114, 39, 16, 251, 205, 147, 174, 152, 198, 69, 38, 171, 115, 120, 45, 64, 254, 21}

$$\begin{aligned}\text{JOHN.3*} = \{&5, 18, 10, 24, 130, 179, 182, 136, 135, 185, 188, 141, 256, 267, 271, 263, 269, 270, 257,\\ &261, 74, 190, 93, 88, 99, 133, 166, 152, 163, 79, 116, 113, 157, 128, 187, 102, 9, 167, 12,\\ &37, 55, 153, 132, 156, 162, 32, 50, 27, 186, 191, 129, 30\}\\ \text{JOHN.4*} = \{&97, 115, 121, 107, 192, 236, 227, 207, 5, 18, 10, 24, 197, 233, 230, 202, 100, 118, 124,\\ &110, 131, 245, 178, 155, 211, 193, 231, 170, 134, 240, 183, 158, 214, 196, 226, 175, 64,\\ &180, 87, 82, 105, 143, 172, 146, 169, 69, 126, 123, 151, 138, 177, 108\}\end{aligned}$$

Using an algorithm similar to the one applied to the designs associated with maximal $(120, 8)$-arcs, we computed all parallel classes, resolutions, and maximals sets of 52 pairwise compatible resolutions for each of the 2-$(52, 4, 1)$ designs associated with maximal $(52, 4)$-arcs in the known planes of order 16. The results of these computations are summarized in Table 1, where we use the notation from [21] for previously known arcs. An arc with a name ending at $*$ denotes a new arc found in [13]. The arcs PG$(2, 16).1$ and PG$(2, 16).2$ are maximal $(52, 4)$-arcs in the Desarguesian plane PG$(2, 16)$. As shown in [2], up to projective equivalence, PG$(2, 16)$ contains only two maximal $(52, 4)$-arcs, both of Denniston type [12].

Table 1: Maximal (52, 4)-arcs and related 2-(52, 4, 1) designs

Arc	|Aut(D)|	2-rank	Par. Cl.	Res.	Comp. Res.	Plane
PG(2, 16).1	68	41	2329	409	52 (×1)	PG(2, 16)
PG(2, 16).2	408	41	2550	460	52 (×1)	PG(2, 16)
DEMP.1	24	49	250	52	52 (×1)	DEMP
DEMP.2	144	47	543	52	52 (×1)	DEMP
SEMI4.1	96	45	2569	52	52 (×1)	SEMI4
SEMI2.1	24	47	327	52	52 (×1)	SEMI2
SEMI2.2	144	45	1279	55	52 (×1)	SEMI2
LMRH.1	96	47	2265	104	52 (×2)	LMRH, LMRH$^{\perp}$
MATH.1	24	49	291	52	52 (×1)	MATH
MATH.2*	32	46	1729	52	52 (×1)	MATH
MATH.3*	32	47	2401	64	52 (×1)	MATH
MATH.4*	32	46	1665	52	52 (×1)	MATH
MATH.5*	16	47	1233	52	52 (×1)	MATH
MATH.6*	16	48	1329	52	52 (×1)	MATH
MATH.7*	16	48	1125	52	52 (×1)	MATH
HALL.1	24	49	274	52	52 (×1)	HALL
BBH1.1	24	47	330	52	52 (×1)	BBH1
BBH1.2	32	46	2017	136	52 (×2)	BBH1, JOHN
JOWK.1	16	46	1389	52	52 (×1)	JOWK
JOWK.2	32	46	2409	104	52 (×2)	JOWK, JOHN
JOHN.1	32	47	1953	144	52 (×2)	JOHN
JOHN.2	32	47	1953	144	52 (×2)	JOHN
JOHN.3*	32	46	2017	136	52 (×2)	JOHN, BBH1
JOHN.4*	32	46	2409	104	52 (×2)	JOHN, JOWK
DSFP.1	24	47	1045	52	52 (×1)	DSFP

The most interesting outcome of these computations is the following phenomenon that provides knew connections between some of the known projective planes of order 16: there are seven maximal $(52, 4)$-arcs (LMRH.1, BBH1.2, JOWK.2, JOHN.1, JOHN.2, JOHN.3* and JOHN.4*), whose related 2-$(52, 4, 1)$ designs admit two different sets of 52 pairwise compatible resolutions, meeting the bound of Theorem 1.1, and consequently are embeddable in two different planes. The two sets of 52 compatible resolutions of the designs associated with the maximal arcs JOHN.1 and JOHN.2 give rise to two projective planes, both isomorphic to the Johnson plane JOHN. However, in the remaining cases, one of the sets of 52 compatible resolutions gives rise to the original plane, while the second set gives rise to another nonisomorphic plane:

- the 2-(52, 4, 1) design associated with the maximal arc LMRH.1 is embeddable in the Lorimer–Rahilly plane LMRH, as well as in its dual plane;
- the 2-(52, 4, 1) design associated with the maximal arc BBH1.2 is embeddable in the plane BBH1 obtained by Bose–Barlotti derivation [5] of the Hall plane, as well as in a second plane isomorphic to the Johnson plane JOHN;
- the 2-(52, 4, 1) design associated with the maximal arc JOWK.2 is embeddable in the Johnson–Walker plane JOWK, as well as in a second plane isomorphic to the Johnson plane;
- the 2-(52, 4, 1) design associated with the maximal arc JOHN.3* is embeddable in the Johnson plane JOHN, as well in a second plane isomorphic to the plane BBH1;
- the 2-(52, 4, 1) design associated with the maximal arc JOHN.4* is embeddable in the Johnson plane JOHN, as well in a second plane isomorphic to the Johnson–Walker plane JOWK.

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