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Calculation of eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument which contains a spectral parameter in the boundary condition

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ABSTRACT

In this work, a discontinuous boundary-value problem with retarded argument which contains a spectral parameter in the boundary condition and with transmission conditions at the point of discontinuity is investigated. We obtained asymptotic formulas for the eigenvalues and eigenfunctions.

1. Introduction

Boundary value problems for differential equations of the second order with retarded argument were studied in [1–5], and various physical applications of such problems can be found in [2].

The asymptotic formulas for the eigenvalues and eigenfunctions of the boundary problem of Sturm–Liouville type for the second order differential equation with retarded argument were obtained in [5].

The asymptotic formulas for the eigenvalues and eigenfunctions of the Sturm–Liouville problem with the spectral parameter in the boundary condition were obtained in [6].

In this paper, we study the eigenvalues and eigenfunctions of the discontinuous boundary value problem with retarded argument and a spectral parameter in the boundary condition. Namely, we consider the boundary value problem for the differential equation

$$p(x)y''(x) + q(x)y(x - \Delta(x)) + \lambda y(x) = 0 \tag{1}$$

on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, with boundary conditions

$$a_1 y(0) + a_2 y'(0) = 0, \tag{2}$$

$$y'(\pi) + \mathrm{d}\lambda y(\pi) = 0, \tag{3}$$

and transmission conditions

$$\gamma_1 y\left(\frac{\pi}{2} - 0\right) = \delta_1 y\left(\frac{\pi}{2} + 0\right), \tag{4}$$

$$\gamma_2 y'\left(\frac{\pi}{2} - 0\right) = \delta_2 y'\left(\frac{\pi}{2} + 0\right), \tag{5}$$

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where $p(x) = p_1^2$ if $x \in \left[0, \frac{\pi}{2}\right)$ and $p(x) = p_2^2$ if $x \in \left(\frac{\pi}{2}, \pi\right]$, the real-valued function $q(x)$ is continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $q\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} q(x)$, the real valued function $\Delta(x) \geq 0$ is continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $\Delta\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} \Delta(x)$, $x - \Delta(x) \geq 0$, if $x \in \left[0, \frac{\pi}{2}\right)$; $x - \Delta(x) \geq \frac{\pi}{2}$, if $x \in \left(\frac{\pi}{2}, \pi\right]$; λ is a real spectral parameter; $p_1, p_2, \gamma_1, \gamma_2, \delta_1, \delta_2, a_1, a_2$, d are arbitrary real numbers; $|a_1| + |a_2| \neq 0$ and $|\gamma_i| + |\delta_i| \neq 0$ for $i = 1, 2$. Also $\gamma_1 \delta_2 p_1 = \gamma_2 \delta_1 p_2$ holds.

It must be noted that some problems with transmission conditions which arise in mechanics (thermal condition problem for a thin laminated plate) were studied in [7].

Let $w_1(x, \lambda)$ be a solution of Eq. (1) on $\left[0, \frac{\pi}{2}\right]$, satisfying the initial conditions

$$w_1(0, \lambda) = a_2, \qquad w_1'(0, \lambda) = -a_1. \tag{6}$$

The conditions (6) define a unique solution of Eq. (1) on $\left[0, \frac{\pi}{2}\right]$ ([2, p. 12]).

After defining the above solution we shall define the solution $w_2(x, \lambda)$ of Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$ by means of the solution $w_1(x, \lambda)$ by the initial conditions

$$w_2\left(\frac{\pi}{2}, \lambda\right) = \gamma_1 \delta_1^{-1} w_1\left(\frac{\pi}{2}, \lambda\right), \qquad \omega_2'\left(\frac{\pi}{2}, \lambda\right) = \gamma_2 \delta_2^{-1} \omega_1'\left(\frac{\pi}{2}, \lambda\right). \tag{7}$$

The conditions (7) are defined as a unique solution of Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$.

Consequently, the function $w(x, \lambda)$ is defined on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ by the equality

$$w(x, \lambda) = \begin{cases} \omega_1(x, \lambda), & x \in \left[0, \dfrac{\pi}{2}\right) \\ \omega_2(x, \lambda), & x \in \left(\dfrac{\pi}{2}, \pi\right] \end{cases}$$

is such a solution of Eq. (1) on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, which satisfies one of the boundary conditions and both transmission conditions.

Lemma 1. *Let $w(x, \lambda)$ be a solution of Eq.* (1) *and $\lambda > 0$. Then the following integral equations hold:*

$$w_1(x, \lambda) = a_2 \cos \frac{s}{p_1} x - \frac{a_1 p_1}{s} \sin \frac{s}{p_1} x - \frac{1}{s} \int_0^x \frac{q(\tau)}{p_1} \sin \frac{s}{p_1} (x - \tau)\, w_1(\tau - \Delta(\tau), \lambda)\, d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right), \tag{8}$$

$$w_2(x, \lambda) = \frac{\gamma_1}{\delta_1} w_1\left(\frac{\pi}{2}, \lambda\right) \cos \frac{s}{p_2}\left(x - \frac{\pi}{2}\right) + \frac{\gamma_2 p_2 w_1'\left(\frac{\pi}{2}, \lambda\right)}{s \delta_2} \sin \frac{s}{p_2}\left(x - \frac{\pi}{2}\right)$$
$$- \frac{1}{s} \int_{\pi/2}^x \frac{q(\tau)}{p_2} \sin \frac{s}{p_2}(x - \tau)\, w_2(\tau - \Delta(\tau), \lambda)\, d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right). \tag{9}$$

Proof. To prove this, it is enough to substitute $-\frac{s^2}{p_1^2} \omega_1(\tau, \lambda) - \omega_1''(\tau, \lambda)$ and $-\frac{s^2}{p_2^2} \omega_2(\tau, \lambda) - \omega_2''(\tau, \lambda)$ instead of $-\frac{q(\tau)}{p_1^2} \omega_1(\tau - \Delta(\tau), \lambda)$ and $-\frac{q(\tau)}{p_2^2} \omega_2(\tau - \Delta(\tau), \lambda)$ in the integrals in (8) and (9) respectively and integrate by parts twice. □

Theorem 1. *The problem* (1)–(5) *can have only simple eigenvalues.*

Proof. Let $\widetilde{\lambda}$ be an eigenvalue of the problem (1)–(5) and

$$\widetilde{u}(x, \widetilde{\lambda}) = \begin{cases} \widetilde{u}_1(x, \widetilde{\lambda}), & x \in \left[0, \dfrac{\pi}{2}\right), \\ \widetilde{u}_2(x, \widetilde{\lambda}), & x \in \left(\dfrac{\pi}{2}, \pi\right] \end{cases}$$

be a corresponding eigenfunction. Then from (2) and (6) it follows that the determinant

$$W\left[\widetilde{u}_1(0, \widetilde{\lambda}), w_1(0, \widetilde{\lambda})\right] = \begin{vmatrix} \widetilde{u}_1(0, \widetilde{\lambda}) & a_2 \\ \widetilde{u}_1(0, \widetilde{\lambda}) & -a_1 \end{vmatrix} = 0,$$

and by Theorem 2.2.2 in [2], the functions $\widetilde{u}_1(x, \widetilde{\lambda})$ and $w_1(x, \widetilde{\lambda})$ are linearly dependent on $\left[0, \frac{\pi}{2}\right]$. We can also prove that the functions $\widetilde{u}_2(x, \widetilde{\lambda})$ and $w_2(x, \widetilde{\lambda})$ are linearly dependent on $\left[\frac{\pi}{2}, \pi\right]$. Hence

$$\widetilde{u}_1(x, \widetilde{\lambda}) = K_i w_i(x, \widetilde{\lambda}) \quad (i = 1, 2) \tag{10}$$

for some $K_1 \neq 0$ and $K_2 \neq 0$. We must show that $K_1 = K_2$. Suppose that $K_1 \neq K_2$. From the equalities (4) and (10), we have

$$\begin{aligned}\gamma_1 \widetilde{u}\left(\frac{\pi}{2}-0, \widetilde{\lambda}\right) - \delta_1 \widetilde{u}\left(\frac{\pi}{2}+0, \widetilde{\lambda}\right) &= \gamma_1 \widetilde{u_1}\left(\frac{\pi}{2}, \widetilde{\lambda}\right) - \delta_1 \widetilde{u_2}\left(\frac{\pi}{2}, \widetilde{\lambda}\right) \\ &= \gamma_1 K_1 w_1\left(\frac{\pi}{2}, \widetilde{\lambda}\right) - \delta_1 K_2 w_2\left(\frac{\pi}{2}, \widetilde{\lambda}\right) \\ &= \gamma_1 K_1 \delta_1 \gamma_1^{-1} w_2\left(\frac{\pi}{2}, \widetilde{\lambda}\right) - \delta_1 K_2 w_2\left(\frac{\pi}{2}, \widetilde{\lambda}\right) \\ &= \delta_1 \left(K_1 - K_2\right) w_2\left(\frac{\pi}{2}, \widetilde{\lambda}\right).\end{aligned}$$

Since $\delta_1 \left(K_1 - K_2\right) \neq 0$ it follows that

$$w_2\left(\frac{\pi}{2}, \widetilde{\lambda}\right) = 0. \tag{11}$$

By the same procedure from equality (5) we can derive that

$$w_2'\left(\frac{\pi}{2}, \widetilde{\lambda}\right) = 0. \tag{12}$$

From the fact that $w_2(x, \widetilde{\lambda})$ is a solution of the differential equation (1) on $\left[\frac{\pi}{2}, \pi\right]$ and satisfies the initial conditions (11) and (12) it follows that $w_1(x, \widetilde{\lambda}) = 0$ identically on $\left[\frac{\pi}{2}, \pi\right]$ (cf. [2, p. 12, Theorem 1.2.1]).

By using this, we may also find

$$w_1\left(\frac{\pi}{2}, \widetilde{\lambda}\right) = w_1'\left(\frac{\pi}{2}, \widetilde{\lambda}\right) = 0.$$

From the latter discussions of $w_2(x, \widetilde{\lambda})$ it follows that $w_1(x, \widetilde{\lambda}) = 0$ identically on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. But this contradicts (6), thus completing the proof. □

2. An existence theorem

The function $\omega(x, \lambda)$ defined in Section 1 is a nontrivial solution of Eq. (1) satisfying conditions (2), (4) and (5). Putting $\omega(x, \lambda)$ into (3), we get the characteristic equation

$$F(\lambda) \equiv \omega'(\pi, \lambda) + d\lambda\omega(\pi, \lambda) = 0. \tag{13}$$

By Theorem 1 the set of eigenvalues of the boundary-value problem (1)–(5) coincides with the set of real roots of Eq. (13). Let $q_1 = \frac{1}{p_1}\int_0^{\pi/2} |q(\tau)| d\tau$ and $q_2 = \frac{1}{p_2}\int_{\pi/2}^{\pi} q(\tau) d\tau$.

Lemma 2. (1) *Let $\lambda \geq 4q_1^2$. Then for the solution $w_1(x, \lambda)$ of Eq. (8), the following inequality holds:*

$$|w_1(x, \lambda)| \leq \frac{1}{q_1}\sqrt{4q_1^2 a_2^2 + p_1^2 a_1^2}, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{14}$$

(2) *Let $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$. Then for the solution $w_2(x, \lambda)$ of Eq. (9), the following inequality holds:*

$$|w_2(x, \lambda)| \leq \frac{2}{q_1}\sqrt{4q_1^2 a_2^2 + p_1^2 a_1^2}\left\{\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_1}\right|\right\}, \quad x \in \left[\frac{\pi}{2}, \pi\right]. \tag{15}$$

Proof. Let $B_{1\lambda} = \max_{[0, \frac{\pi}{2}]} |w_1(x, \lambda)|$. Then from (8), it follows that, for every $\lambda > 0$, the following inequality holds:

$$B_{1\lambda} \leq \sqrt{a_2^2 + \frac{p_1^2 a_1^2}{s^2}} + \frac{1}{s} B_{1\lambda} q_1.$$

If $s \geq 2q_1$ we get (14). Differentiating (8) with respect to x, we have

$$w_1'(x, \lambda) = -\frac{s a_2}{p_1}\sin\frac{s}{p_1}x - a_1\cos\frac{s}{p_1}x - \frac{1}{p_1^2}\int_0^x q(\tau)\cos\frac{s}{p_1}(x-\tau)\, w_1(\tau - \Delta(\tau)) d\tau. \tag{16}$$

From (16) and (14), it follows that, for $s \geq 2q_1$, the following inequality holds:

$$\left|w_1'(x, \lambda)\right| \leq \sqrt{\frac{s^2 a_2^2}{p_1^2} + a_1^2} + \frac{1}{p_1}\sqrt{4q_1^2 a_2^2 + p_1^2 a_1^2}.$$

Hence

$$\frac{\left|w_1'(x,\lambda)\right|}{s} \leq \frac{1}{p_1 q_1}\sqrt{4q_1^2 a_2^2 + p_1^2 a_1^2}. \tag{17}$$

Let $B_{2\lambda} = \max_{[\frac{\pi}{2},\pi]} |w_2(x,\lambda)|$. Then from (9), (14) and (17) it follows that, for $s \geq 2q_1$, the following inequalities hold:

$$B_{2\lambda} \leq \frac{1}{q_1}\left|\frac{\gamma_1}{\delta_1}\right|\sqrt{4q_1^2a_2^2+p_1^2a_1^2} + |p_2|\left|\frac{\gamma_2}{\delta_2}\right|\frac{1}{|p_1q_1|}\sqrt{4q_1^2a_2^2+p_1^2a_1^2} + \frac{1}{2q_2}B_{2\lambda}q_2,$$

$$B_{2\lambda} \leq \frac{2}{q_1}\sqrt{4q_1^2a_2^2+p_1^2a_1^2}\left\{\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_1}\right|\right\}.$$

Hence if $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$ we get (15). □

Theorem 2. *The problem (1)–(5) has an infinite set of positive eigenvalues.*

Proof. Differentiating (9) with respect to x, we get

$$w_2'(x,\lambda) = -\frac{s\gamma_1}{p_2\delta_1}w_1'\left(\frac{\pi}{2},\lambda\right)\sin\frac{s}{p_2}\left(x-\frac{\pi}{2}\right) + \frac{\gamma_2 w_1'\left(\frac{\pi}{2},\lambda\right)}{\delta_2}\cos\frac{s}{p_2}\left(x-\frac{\pi}{2}\right)$$
$$-\frac{1}{p_2^2}\int_{\pi/2}^{x} q(\tau)\cos\frac{s}{p_2}(x-\tau)\,w_2(\tau-\Delta(\tau),\lambda)\mathrm{d}\tau. \tag{18}$$

From (8), (9), (13), (16) and (18), we get

$$-\frac{s\gamma_1}{p_2\delta_1}\left(a_2\cos\frac{s\pi}{2p_1} - \frac{a_1}{s}\sin\frac{s\pi}{2p_1} - \frac{1}{sp_1}\int_0^{\frac{\pi}{2}} q(\tau)\sin\frac{s}{p_1}\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)\mathrm{d}\tau\right)\sin\frac{s\pi}{2p_2}$$
$$+\frac{\gamma_2}{\delta_2}\left(-\frac{sa_2}{p_1}\sin\frac{s\pi}{2p_1} - a_1\cos\frac{s\pi}{2p_1} - \frac{1}{p_1^2}\int_0^{\frac{\pi}{2}} q(\tau)\cos\frac{s}{p_1}\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)\mathrm{d}\tau\right)$$
$$\times\cos\frac{s\pi}{2p_2} - \frac{1}{p_2^2}\int_{\pi/2}^{\pi} q(\tau)\cos\frac{s}{p_2}(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)\mathrm{d}\tau$$
$$+\lambda d\left(\frac{\gamma_1}{\delta_1}\left[a_2\cos\frac{s\pi}{2p_1} - \frac{a_1p_1}{s}\sin\frac{s\pi}{2p_1} - \frac{1}{sp_1}\int_0^{\frac{\pi}{2}} q(\tau)\sin\frac{s}{p_1}\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)\mathrm{d}\tau\right]\cos\frac{s\pi}{2p_2}\right.$$
$$+\frac{\gamma_2p_2}{\delta_2 s}\left[-\frac{sa_2}{p_1}\sin\frac{s\pi}{2p_1} - a_1\cos\frac{s\pi}{2p_1} - \frac{1}{p_1^2}\int_0^{\frac{\pi}{2}} q(\tau)\cos\frac{s}{p_1}\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)\mathrm{d}\tau\right]$$
$$\left.\times\sin\frac{s\pi}{2p_2} - \frac{1}{sp_2}\int_{\frac{\pi}{2}}^{\pi} q(\tau)\sin\frac{s}{p_2}(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)\mathrm{d}\tau\right). \tag{19}$$

There are two possible cases: (1) $a_2 \neq 0$, (2) $a_2 = 0$. In this paper, we shall consider only case (1). The other cases may be considered analogically. Let λ be sufficiently large. Then, by (14) and (15), Eq. (19) may be rewritten in the form

$$s\cos s\pi\frac{p_1+p_2}{2p_1p_2} + O(1) = 0. \tag{20}$$

Obviously, for large s Eq. (20) has an infinite set of roots. Thus the theorem is proved. □

3. Asymptotic formulas for eigenvalues and eigenfunctions

Now we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following we shall assume that s is sufficiently large. From (8) and (14), we get

$$\omega_1(x,\lambda) = O(1) \quad \text{on } \left[0,\frac{\pi}{2}\right]. \tag{21}$$

From (9) and (15), we get

$$\omega_2(x,\lambda) = O(1) \quad \text{on } \left[\frac{\pi}{2},\pi\right]. \tag{22}$$

The existence and continuity of the derivatives $\omega'_{1s}(x,\lambda)$ for $0 \le x \le \frac{\pi}{2}$, $|\lambda| < \infty$, and $\omega'_{2s}(x,\lambda)$ for $\frac{\pi}{2} \le x \le \pi$, $|\lambda| < \infty$, follow from Theorem 1.4.1 in [2].

$$\omega'_{1s}(x,\lambda) = O(1), \quad x \in \left[0, \frac{\pi}{2}\right] \quad \text{and} \quad \omega'_{2s}(x,\lambda) = O(1), \quad x \in \left[\frac{\pi}{2}, \pi\right]. \tag{23}$$

Theorem 3. *Let n be a natural number. For each sufficiently large n, in case (1), there is exactly one eigenvalue of the problem (1)–(5) near $\frac{p_1^2 p_2^2}{(p_1+p_2^2)}(2n+1)^2$.*

Proof. We consider the expression which is denoted by $O(1)$ in Eq. (20). If formulas (21)–(23) are taken into consideration, it can be shown by differentiation with respect to s that for large s this expression has bounded derivative. It is obvious that for large s the roots of Eq. (20) are situated close to entire numbers. We shall show that, for large n, only one root (20) lies near to each $\frac{p_1^2 p_2^2}{(p_1+p_2)^2}(2n+1)^2$. We consider the function $\phi(s) = s\cos s\pi\frac{p_1+p_2}{2p_1p_2} + O(1)$. Its derivative, which has the form $\phi'(s) = \cos s\pi\frac{p_1+p_2}{2p_1p_2} - s\pi\frac{p_1+p_2}{2p_1p_2}\sin s\pi\frac{p_1+p_2}{2p_1p_2} + O(1)$, does not vanish for s close to n for sufficiently large n. Thus our assertion follows by Rolle's theorem. □

Let n be sufficiently large. In what follows we shall denote by $\lambda_n = s_n^2$ the eigenvalue of the problem (1)–(5) situated near $\frac{p_1^2 p_2^2}{(p_1+p_2)^2}(2n+1)^2$. We set $s_n = \frac{p_1p_2(2n+1)}{p_1+p_2} + \delta_n$. From (20) it follows that $\delta_n = O\left(\frac{1}{n}\right)$. Consequently

$$s_n = \frac{p_1p_2\,(2n+1)}{p_1+p_2} + O\left(\frac{1}{n}\right). \tag{24}$$

The formula (24) makes it possible to obtain asymptotic expressions for the eigenfunction of the problem (1)–(5). From (8), (16) and (21), we get

$$\omega_1(x,\lambda) = a_2\cos\frac{s}{p_1}x + O\left(\frac{1}{s}\right), \tag{25}$$

$$\omega_1'(x,\lambda) = -\frac{sa_2}{p_1}\sin\frac{s}{p_1}x + O\,(1)\,. \tag{26}$$

From (9), (22), (25) and (26), we get

$$\omega_2(x,\lambda) = \frac{\gamma_1 a_2}{\delta_1}\cos s\left(\frac{\pi\,(p_2 - p_1)}{2p_1p_2} + \frac{x}{p_2}\right) + O\left(\frac{1}{s}\right). \tag{27}$$

By putting (24) in (25) and (27), we derive that

$$u_{1n} = w_1\,(x,\lambda_n) = a_2\cos\frac{p_2\,(2n+1)}{p_1+p_2}x + O\left(\frac{1}{n}\right),$$

$$u_{2n} = w_2\,(x,\lambda_n) = \frac{\gamma_1 a_2}{\delta_1}\cos\left(\frac{\pi\,(p_2-p_1)\,(2n+1)}{2\,(p_1+p_2)} + \frac{p_1\,(2n+1)}{p_1+p_2}x\right) + O\left(\frac{1}{n}\right).$$

Hence the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x) = \begin{cases} a_2\cos\dfrac{p_2\,(2n+1)}{p_1+p_2}x + O\left(\dfrac{1}{n}\right) & \text{for } x \in \left[0, \dfrac{\pi}{2}\right), \\ \dfrac{\gamma_1}{\delta_1}\cos\left(\dfrac{\pi\,(p_2-p_1)\,(2n+1)}{2\,(p_1+p_2)} + \dfrac{p_1\,(2n+1)}{p_1+p_2}x\right) + O\left(\dfrac{1}{n}\right) & \text{for } x \in \left(\dfrac{\pi}{2}, \pi\right]. \end{cases}$$

Under some additional conditions, more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

(a) The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $\left[0, \frac{\pi}{2}\right) \bigcup \left(\frac{\pi}{2}, \pi\right]$ and have finite limits $q'\left(\frac{\pi}{2} \pm 0\right) = \lim_{x\to\frac{\pi}{2}\pm 0} q'(x)$ and $\Delta''\left(\frac{\pi}{2} \pm 0\right) = \lim_{x\to\frac{\pi}{2}\pm 0}\Delta''(x)$, respectively.

(b) $\Delta'(x) \le 1$ in $\left[0, \frac{\pi}{2}\right) \bigcup \left(\frac{\pi}{2}, \pi\right]$, $\Delta(0) = 0$ and $\lim_{x\to\frac{\pi}{2}+0}\Delta(x) = 0$.

By using (b), we have

$$x - \Delta(x) \ge 0, \quad \text{for } x \in \left[0, \frac{\pi}{2}\right) \quad \text{and} \quad x - \Delta(x) \ge \frac{\pi}{2}, \quad \text{for } x \in \left(\frac{\pi}{2}, \pi\right]. \tag{28}$$

From (25), (27) and (28) we have

$$w_1(\tau-\Delta(\tau),\lambda)=a_2\cos\frac{s}{p_1}(\tau-\Delta(\tau))+O\left(\frac{1}{s}\right), \tag{29}$$

$$w_2(\tau-\Delta(\tau),\lambda)=\frac{\gamma_1 a_2}{\delta_1}\cos s\left(\frac{\pi(p_2-p_1)}{2p_1p_2}+\frac{\tau-\Delta(\tau)}{p_2}\right)+O\left(\frac{1}{s}\right). \tag{30}$$

Putting these expressions into (19), we have

$$\begin{aligned}
0=&\ \frac{s\mathrm{d}a_2\gamma_1}{\delta_1}\cos\frac{s\pi(p_1+p_2)}{2p_1p_2}-\frac{\gamma_1}{\delta_1}\left(\mathrm{d}a_1p_1+\frac{a_2}{p_2}\right)\sin\frac{s\pi(p_1+p_2)}{2p_1p_2}\\
&-\frac{\mathrm{d}a_2\gamma_1}{\delta_1}\left[\frac{1}{p_1}\sin\frac{s\pi(p_1+p_2)}{2p_1p_2}\int_0^{\pi/2}\frac{q(\tau)}{2}\left[\cos\frac{s\Delta(\tau)}{p_1}+\cos\frac{s}{p_1}(2\tau-\Delta(\tau))\right]\mathrm{d}\tau\right.\\
&-\frac{1}{p_1}\cos\frac{s\pi(p_1+p_2)}{2p_1p_2}\int_0^{\pi/2}\frac{q(\tau)}{2}\left[\sin\frac{s\Delta(\tau)}{p_1}+\sin\frac{s}{p_1}(2\tau-\Delta(\tau))\right]\mathrm{d}\tau\\
&+\frac{1}{p_2}\cos\frac{s\pi(p_2-p_1)}{2p_1p_2}\sin\frac{s\pi}{p_2}\int_{\pi/2}^{\pi}\frac{q(\tau)}{2}\left[\cos\frac{s\Delta(\tau)}{p_2}+\cos\frac{s}{p_2}(2\tau-\Delta(\tau))\right]\mathrm{d}\tau\\
&-\frac{1}{p_2}\cos\frac{s\pi(p_2-p_1)}{2p_1p_2}\cos\frac{s\pi}{p_2}\int_{\pi/2}^{\pi}\frac{q(\tau)}{2}\left[\sin\frac{s\Delta(\tau)}{p_2}+\sin\frac{s}{p_2}(2\tau-\Delta(\tau))\right]\mathrm{d}\tau\\
&-\frac{1}{p_2}\sin\frac{s\pi(p_2-p_1)}{2p_1p_2}\sin\frac{s\pi}{p_2}\int_{\pi/2}^{\pi}\frac{q(\tau)}{2}\left[\sin\frac{s\Delta(\tau)}{p_2}-\sin\frac{s}{p_2}(2\tau-\Delta(\tau))\right]\mathrm{d}\tau\\
&\left.-\frac{1}{p_2}\sin\frac{s\pi(p_2-p_1)}{2p_1p_2}\cos\frac{s\pi}{p_2}\int_{\pi/2}^{\pi}\frac{q(\tau)}{2}\left[\cos\frac{s\Delta(\tau)}{p_2}-\cos\frac{s}{p_2}(2\tau-\Delta(\tau))\right]\mathrm{d}\tau\right]+O\left(\frac{1}{s}\right).
\end{aligned} \tag{31}$$

Let

$$A(x,s,\Delta(\tau))=\frac{1}{2}\int_0^x q(\tau)\sin\frac{s}{p_1}\Delta(\tau)\,\mathrm{d}\tau,\qquad B(x,s,\Delta(\tau))=\frac{1}{2}\int_0^x q(\tau)\cos\frac{s}{p_1}\Delta(\tau)\,\mathrm{d}\tau. \tag{32}$$

It is obvious that these functions are bounded for $0\le x\le\pi$, $0<s<\infty$. Let

$$C(x,s,\Delta(\tau))=\frac{1}{2}\int_{\pi/2}^x q(\tau)\sin\frac{s}{p_2}\Delta(\tau)\,\mathrm{d}\tau,\qquad D(x,s,\Delta(\tau))=\frac{1}{2}\int_{\pi/2}^x q(\tau)\cos\frac{s}{p_2}\Delta(\tau)\,\mathrm{d}\tau. \tag{33}$$

It is obvious that these functions are bounded for $\frac{\pi}{2}\le x\le\pi$, $0<s<\infty$.
Under the conditions (a) and (b) the following formulas

$$\begin{aligned}
&\int_0^x q(\tau)\cos\frac{s}{p_1}(2\tau-\Delta(\tau))\mathrm{d}\tau=O\left(\frac{1}{s}\right),\qquad \int_0^x q(\tau)\sin\frac{s}{p_1}(2\tau-\Delta(\tau))\mathrm{d}\tau=O\left(\frac{1}{s}\right)\\
&\int_{\pi/2}^x q(\tau)\cos\frac{s}{p_2}(2\tau-\Delta(\tau))\mathrm{d}\tau=O\left(\frac{1}{s}\right),\qquad \int_{\pi/2}^x q(\tau)\sin\frac{s}{p_2}(2\tau-\Delta(\tau))\mathrm{d}\tau=O\left(\frac{1}{s}\right)
\end{aligned} \tag{34}$$

can be proved by the same technique in Lemma 3.3.3 in [2]. From (31)–(34) and $s_n=\frac{p_1p_2(2n+1)}{p_1+p_2}+\delta_n$, we have

$$\begin{aligned}
\cot\left(\frac{\pi}{2}(2n+1)+\frac{\pi(p_1+p_2)\delta_n}{2p_1p_2}\right)=&\ \frac{p_1+p_2}{(2n+1)p_1p_2}\left[\frac{\mathrm{d}}{p_2}D\left(\pi,\frac{p_1p_2(2n+1)}{p_1+p_2},\Delta(\tau)\right)\right.\\
&\left.+\frac{\mathrm{d}a_1p_1}{a_2}+\frac{1}{p_2}+\frac{\mathrm{d}}{p_1}B\left(\frac{\pi}{2},\frac{p_1p_2(2n+1)}{p_1+p_2},\Delta(\tau)\right)\right]+O\left(\frac{1}{n^2}\right)
\end{aligned}$$

and finally

$$\begin{aligned}
s_n=&\ \frac{p_1p_2(2n+1)}{p_1+p_2}-\frac{2}{\pi(2n+1)}\left[\frac{\mathrm{d}}{p_2}D\left(\pi,\frac{p_1p_2(2n+1)}{p_1+p_2},\Delta(\tau)\right)\right.\\
&\left.+\frac{\mathrm{d}a_1p_1}{a_2}+\frac{1}{p_2}+\frac{\mathrm{d}}{p_1}B(\frac{\pi}{2},\frac{p_1p_2(2n+1)}{p_1+p_2},\Delta(\tau))\right]+O\left(\frac{1}{n^2}\right).
\end{aligned} \tag{35}$$

Thus, we have proven the following theorem.

Theorem 4. *If conditions* (a) *and* (b) *are satisfied, then the positive eigenvalues* $\lambda_n = s_n^2$ *of the problem* (1)–(5) *have the* (35) *asymptotic representation for* $n \to \infty$.

We now may obtain a sharper asymptotic formula for the eigenfunctions. From (8) and (29)

$$w_1(x,\lambda) = a_2 \cos\frac{s}{p_1}x - \frac{a_1 p_1}{s}\sin\frac{s}{p_1}x - \frac{a_2 p_1}{s}\int_0^x q(\tau)\sin\frac{s}{p_1}(x-\tau)\cos\frac{s}{p_1}(\tau - \Delta(\tau))\,d\tau + O\left(\frac{1}{s^2}\right).$$

Thus, from (32)–(34)

$$w_1(x,\lambda) = a_2\cos\frac{s}{p_1}x\left[1 + \frac{A\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{sp_1}\right] - \frac{\sin\frac{s}{p_1}x}{s}\left[a_1 p_1 + \frac{a_2}{p_1}B(x, s, \Delta(\tau))\right] + O\left(\frac{1}{s^2}\right). \tag{36}$$

Replacing s by s_n and using (35), we have

$$\begin{aligned} u_{1n}(x) &= a_2\cos\frac{p_2(2n+1)}{p_1+p_2}x\left[1 + \frac{(p_1+p_2)A\left(x, \frac{p_1p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)}{p_1p_2(2n+1)}\right] \\ &\quad + a_2\sin\frac{p_2(2n+1)}{p_1+p_2}x\left[\frac{2x}{\pi(2n+1)p_1}\left[\frac{dD\left(\pi, \frac{p_1p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)}{p_2} + \frac{da_1p_1}{a_2}\right.\right. \\ &\quad \left.\left. + \frac{1}{p_2} + dB\left(\frac{\pi}{2}, \frac{p_1p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)\right]\right] - \frac{p_1+p_2}{p_1p_2(2n+1)}\sin\frac{p_2(2n+1)}{p_1+p_2}x \\ &\quad \times\left[a_1p_1 + \frac{a_2B\left(x, \frac{p_1p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)}{p_1}\right] + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{37}$$

From (16), (29) and (32), we have

$$\frac{w_1'(x,\lambda)}{s} = -\frac{a_2}{p_1}\sin\frac{s}{p_1}x\left(1 + \frac{A(x,s,\Delta(\tau))}{sp_1}\right) - \frac{\cos\frac{s}{p_1}x}{s}\left(a_1 + \frac{a_2}{p_1^2}B(x,s,\Delta(\tau))\right) + O\left(\frac{1}{s^2}\right), \quad x\in\left(0, \frac{\pi}{2}\right]. \tag{38}$$

From (9), (30), (34), (36) and (38) we have

$$\begin{aligned} w_2(x,\lambda) &= \frac{\gamma_1}{\delta_1}\left\{a_2\cos\frac{s\pi}{2p_1}\left[1 + \frac{A\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{sp_1}\right] - \frac{\sin\frac{s\pi}{2p_1}}{s}\left[a_1p_1 + \frac{a_2B\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{p_1}\right]\right. \\ &\quad \left. + O\left(\frac{1}{s^2}\right)\right\}\cos\frac{s}{p_2}\left(x - \frac{\pi}{2}\right) - \frac{\gamma_2p_2}{\delta_2p_1}\left\{a_2\sin\frac{s\pi}{2p_1}\left[1 + \frac{A\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{sp_1}\right]\right. \\ &\quad \left. + \frac{\cos\frac{s\pi}{2p_1}}{s}\left[a_1p_1 + \frac{a_2B\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{p_1}\right] + O\left(\frac{1}{s^2}\right)\right\}\sin\frac{s}{p_2}\left(x - \frac{\pi}{2}\right) - \frac{1}{sp_2} \\ &\quad \times\int_{\pi/2}^x q(\tau)\sin\frac{s}{p_2}(x-\tau)\left[\frac{\gamma_1a_2}{\delta_1}\cos\frac{s}{p_2}\left(\frac{\pi(p_2-p_1)}{2p_1} + \tau - \Delta(\tau)\right) + O\left(\frac{1}{s}\right)d\tau\right] \\ &= \left\{\frac{\gamma_1a_2}{\delta_1}\left[1 + \frac{A\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{sp_1}\right] + \frac{\gamma_1a_2C(x,s,\Delta(\tau))}{sp_2\delta_1}\right\}\cos\frac{s}{p_2}\left(x + \frac{\pi(p_2-p_1)}{2p_1}\right) \\ &\quad - \left\{\frac{\gamma_1a_2D(x,s,\Delta(\tau))}{sp_2\delta_1} + \frac{\gamma_1}{s\delta_1}\left[a_1p_1 + \frac{a_2B\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{p_1}\right]\right\} \\ &\quad \times\sin\frac{s}{p_2}\left(x + \frac{\pi(p_2-p_1)}{2p_1}\right) + O\left(\frac{1}{s^2}\right), \quad x\in\left(\frac{\pi}{2}, \pi\right]. \end{aligned}$$

Now, replacing s by s_n and using (35), we have

$$\begin{aligned} u_{2n}(x) = & \left\{ \frac{\gamma_1 a_2}{\delta_1} \left[1 + \frac{(p_1+p_2) A\left(\frac{\pi}{2}, \frac{p_1 p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)}{p_1^2 p_2 (2n+1)} \right] \right. \\ & \left. + \frac{\gamma_1 (p_1+p_2) A\left(x, \frac{p_1 p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)}{\delta_1 p_1 p_2^2 (2n+1)} \right\} \cos\left(\frac{p_1 x(2n+1)}{p_{1+}p_2} + \frac{\pi (p_2-p_1)(2n+1)}{2(p_{1+}p_2)} \right) \\ & \times \left\{ \frac{\gamma_1 a_2}{\delta_1} \left[\frac{2}{\pi (2n+1)} \left[\frac{dA\left(\pi, \frac{p_1 p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)}{p_2} + \frac{da_1 p_1}{a_2} + \frac{1}{p_2} \right. \right. \right. \\ & \left. \left. + \frac{dB\left(\frac{\pi}{2}, \frac{p_1 p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)}{p_1} \right] \right] \left(\frac{x}{p_2} + \frac{\pi (p_2-p_1)}{2p_1 p_2} \right) \\ & - \left[\frac{a_2 \gamma_1 (p_1+p_2) D\left(x, \frac{p_1 p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)}{p_1 p_2^2 \delta_1 (2n+1)} + \frac{\gamma_1 (p_1+p_2)}{p_1 p_2 \delta_1 (2n+1)} \left(a_1 p_1 \right. \right. \\ & \left. \left. \left. + \frac{a_2 B\left(\frac{\pi}{2}, \frac{p_1 p_2(2n+1)}{p_1+p_2}, \Delta(\tau)\right)}{p_1} \right) \right] \right\} \sin\left(\frac{p_1 x(2n+1)}{p_{1+}p_2} + \frac{\pi (p_2-p_1)(2n+1)}{2(p_{1+}p_2)} \right) + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{39}$$

Thus, we have proven the following theorem.

Theorem 5. *If conditions* (a) *and* (b) *are satisfied, then the eigenfunctions $u_n(x)$ of the problem* (1)–(5) *have the following asymptotic representation for $n \to \infty$:*

$$u_n(x) = \begin{cases} u_{1n}(x) & \text{for } x \in \left[0, \frac{\pi}{2}\right) \\ u_{2n}(x) & \text{for } x \in \left(\frac{\pi}{2}, \pi\right] \end{cases}$$

where $u_{1n}(x)$ and $u_{2n}(x)$ defined as in (37) *and* (39)*, respectively.*

4. Conclusion

In this study, first we obtain asymptotic formulas for eigenvalues and eigenfunctions for the discontinuous boundary value problem with retarded argument which contains a spectral parameter in the boundary condition. Then under additional conditions (a) and (b), more exact asymptotic formulas which depend upon the retardation are obtained.

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