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Semi-Analytical Solutions of Nonlinear Equation Modelling Reaction in a Dispersive Medium

Dağılımlı Bir Ortamda Doğrusal Olmayan Reaksiyon Model Denkleminin Yarı Analitik Çözümleri

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Abstract

This study explores the semi-analytical solutions of the third-order dispersive equation with reaction (Fisher-like) term. Recently, the proposed problem has been exactly solved in the literature. Additionally, the semi-analytical solutions are needed to understand the sensitivity of homotopy based methods in solving the proposed reaction-dispersion equation. Using symbolic computation with carefully chosen perturbation parameters, the semi-analytical solutions are compared with the exact solutions, in order to show the efficiency of homotopy and Padé techniques. Obtained solutions, which can play key role in modelling reaction in a dispersive medium, are illustrated and discussed.

Keywords: Homotopy analysis method, Padé approximation, Reaction-dispersion equation, Semi-analytical solutions

Öz

Bu çalışma, üçüncü mertebeden reaksiyon terimli dağılım(dispersive) denkleminin yarı analitik çözümlerini üzerinedir. Son zamanlarda ele alınan problem literatürde tam olarak çözülmüştür. Ayrıca, yarı analitik çözümler, önerilen reaksiyon-dağılım denkleminin çözümünde homotopi temelli yöntemlerin hassasiyetini anlamak için gereklidir. Seçilen pertürbasyon parametreleri ile sembolik hesaplama kullanarak, yarı analitik çözümler, homotopi ve Padé tekniklerinin verimliliğini göstermek için kesin çözümlerle karşılaştırılmaktadır. Elde edilen çözümler dağılımlı ortamda reaksiyon modellemesinde büyük rol oynamaktadır.

Anahtar Kelimeler: Homotopi analiz metodu, Padé yaklaşımı, Reaksiyon-dağılım denklemi, Yarı-analitik çözümler

1. Introduction

There have been many analytical and numerical studies on the equations modelling reaction in the diffusion medium (reaction-diffusion equations), such as the Fisher's equation (or KPP (Kolmogorov-Petrovskii-Piskunov) equation) (Fisher 1937, Kolmogorov et al. 1937, Wazwaz 2008).

$$u_t = u_{xx} + u(1-u), \tag{1}$$

which has travelling solutions (TWs) as follows:

$$u_{1,2}(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\frac{5}{12}t \pm \frac{\sqrt{6}}{12}(x+x_0) + \frac{1}{4}\tanh\frac{5}{12}t \pm \frac{\sqrt{6}}{12}(x+x_0)^2. \tag{2}$$

On the other hand, recently, TWs of newly-proposed dispersive-Fisher equation,

$$u_t = u_{xxx} + u(1-u), \tag{3}$$

which can be used in modelling reaction in the dispersive medium, are revealed in (Pinar and Kocak 2018), (Kocak and Pinar 2017) as

$$u_1(x,t) = \frac{1}{2} - \frac{9}{4}\tanh\frac{19}{60}t - \frac{1}{60}9900^{1/3}(x+x_0) + \frac{11}{4}\tanh\frac{19}{60}t - \frac{1}{60}9900^{1/3}(x+x_0)^3, \tag{4}$$

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$$u_2(x,t)=\frac{1}{2}+\frac{3}{4}\tanh\frac{19}{60}t+\frac{1}{60}30^{2/3}(x+x_0)-\frac{1}{4}\tanh\frac{19}{60}t+\frac{1}{60}30^{2/3}(x+x_0)^3, \tag{5}$$

which are intriguing for further numerical and analytical studies compared to equation (2). Therefore, our aim is to give first attempt on revealing the semi-analytical solutions of equation (3) using homotopy and Padé approaches.

The homotopy methods transform a nonlinear problem to a simple problem, which then can be easily solved (He 2004, Liao 1992). In this study, the Homotopy Analysis Method (HAM) and the Homotopy Analysis Padé Method (HAPM) detailed in the next section are applied to equation (3), in order to obtain semi-analytical solutions.

2. Methodology

Analytical solutions are exceptional in many branches of engineering and physics because of singularities, nonlinearities, inhomogeneity and some general initial-boundary conditions. Hence, applied mathematicians, engineers and physicists are forced to find semi-analytical solutions of the problems in hand. The most common semi-analytical solutions in the literature involve various asymptotic expansions. Here, the asymptotic expansions may need additional special mathematical tools to be convergent, in contrast to power series. But there are practical problems associated with this. First, some asymptotic expansions may converge very slowly. Second, even if enough terms are taken in the expansions, the summing of the expansions usually gives rise to round-off error. Therefore, in recent years, there some novel methods based on integral expansion of the solution before the asymptotic expansion, such as homotopy analysis method (HAM), homotopy analysis Padé method (HAPM) and homotopy perturbation method (HPM). More recently, optimal homotopy asymptotic method (OHAM) and homotopy analysis transform method (HATM), which are based on the same homotopy theory, are proposed. Although, all these methods are known, let us briefly mention the methodology given in (Liao 1992, Liao, 1999, Liao, 2003, Liao, 2012).

Now, for the aforementioned methods: consider a general nonlinear differential equation

$$N[u(x)]=0, \tag{6}$$

where N is a nonlinear operator. Assume that L is an auxiliary linear operator, $u_0(x)=u(x,0)$ is an initial approximation for $u(x,t)$, and h is a constant called the convergence-control parameter.

From the homotopy theory, one can build up a family of equations by using the embedding parameter $p\in[0,1]$, i.e.

$$(1-p)L[U(x;p)-u_0(x;p)]=hpN[U(x;p)], \tag{7}$$

which is known as the zeroth-order deformation equation depending on the embedding parameter $p\in[0,1]$.

Here, the equation,

$$L[U(x;p)-u_0(x;p)]=0,$$

is linear with known initial approximation $U(x;0)=u_0(x)$ when p = 0, but corresponds to $N[u(x)]=0$ when p = 1, hence $U(x;1)=u(x)$. Thus, as p changes from 0 to 1, the solution $U(x;p)$ of the zeroth-order deformation transform the chosen initial approximation $u_0(x)$ to $u(x,t)$ of the proposed nonlinear equation.

In HAM, Liao (1992) expands $U(x;p)$ in a Taylor series about p = 0 to attain homotopy-Taylor series (i.e. Maclaurin series) which reads as:

$$U(x;p)=u_0(x)+\sum_{m=1}^{\infty}u_m(x)p^m \tag{8}$$

and determines h convergence-control parameter of equation (7), wherein equation (8) is convergent for p = 1. Thus the homotopy-Taylor series reduces to the desired solution as follows:

$$U(x)=u_0(x)+\sum_{m=1}^{\infty}u_m(x). \tag{9}$$

In a similar manner, by using equation (8), the m^{th} order deformation equation for $u^m(x)$ can easily be derived. The determination of high-order deformations rather cumbersome manually but can be determined easily by using algebraic computation software. Here, the proposed nonlinear equation is deformed to a set of linear equations, wherein the solution series can be easily obtained.

Later Marinca and Herişanu (2008) modified the classical HAM by introducing a variable convergence-control parameter using a so-called auxiliary function $h(p)$ in the form of

$$h(p)=pC_1+p^2C_2+..., \tag{10}$$

where constants $C_1,C_{2,...}$ are to be determined later, and they consider the solution series (9) including these new parameters $C_i, i=1,2,...$

$$U(x;p,C_i)=u_0(x)+\sum_{m=1}^{\infty}u_m(x,C_i)p^m. \tag{11}$$

Hence, the auxiliary constants $C_1,C_{2,...}$ effect the convergence of the series (11). If it is convergent on p = 1 and one can determine the approximate sets of solution of Equation (8) as:

$$U^{(k)}(x;C_i) = u_0(x) + \sum_{m=1}^{k} u_m(x,C_i), i = 1,2,...,k. \tag{12}$$

Hence, the best approximation (or optimal error) is obtained by optimizing the residual $R(x,C_i)$, therefore theoretically, if $R(x,C_i) = 0$ then $U^{(k)}(x;C_i)$ happens to be exact solution.

On the other hand, He (1998) re-wrote deformation Equation (7) without convergence-control parameter *h* as:

$$(1-p)L[U(x;p) - u_0(x;p)] = pN[U(x;p)], \tag{13}$$

and instead using homotopy-Taylor series (i.e. Eq. 8), He used perturbation series

$$U = u_0 + pu_1 + p^2 u_2 + ..., \tag{14}$$

where using the limit as $p \to 1$ yields the following approximate solution:

$$u = \lim_{p \to 1} U = u_0 + u_1 u_2 + ... \tag{15}$$

Here, the convergence of the series (Eq. 14) is proven in (He 1999).

Nowadays, the modification of HAM namely, HAPM couples the homotopy analysis method and the Padé approximation (Brezenski 1996), (Baker et al. 1996). In this approach, Padé approximation is used to make the expansion convergent even for large values of time, *t*.

3. Model Equation

In this part, we obtain the semi-analytical solutions of dispersive-Fisher equation (3) by using the HAM and HAPM. For equation (3), the HAM procedure can be given in the following form:

$$\begin{aligned} u_n(x,t) &= (1+h)u_{n-1}(x,t) - (1+h)u_{n-1}(x,0) - \\ h\int &(u_{n-1}(x,t)_{xxx} + u_{n-1}(x,t)(1-u_{n-1}(x,t)))dt. \end{aligned} \tag{16}$$

3.1 Numerical Illustration

Example 1. Let us first consider equation (3) with the initial condition

$$\begin{aligned} u_0(x) = u(x,0) = \frac{1}{2} - \frac{9}{4}\tanh\frac{1}{60}9900^{1/3}x + \\ \frac{11}{4}\tanh\frac{1}{60}9900^{1/3}x^3. \end{aligned} \tag{17}$$

Using procedure (16) with the initial condition (17) for equation (3), we can obtain successively,

$$u_1(x,t) = -\frac{19ht\ 8\cosh\frac{1}{60}9900^{1/3}x^2 - 11}{\cosh\frac{1}{60}9900^{1/3}x^4}$$

$$u_2(x,t) = \frac{19}{2400\cosh\frac{1}{60}9900^{1/3}x^5}$$

$$-240\cosh\frac{1}{60}9900^{1/3}x^3 + 330\cosh\frac{1}{60}9900^{1/3}x$$

$$ht - 240\cosh\frac{1}{60}9900^{1/3}x^3 h + 330h\cosh\frac{1}{60}9900^{1/3}x$$

$$+76ht\cosh\frac{1}{60}9900^{1/3}x^2\sinh\frac{1}{60}9900^{1/3}x - 209ht\sinh\frac{1}{60}9900^{1/3}x$$

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and so on. By taking $n = 12$, i.e. finding the 12-th order approximate solution, first we need to determine optimum value of *h* given by the Figure 1. Once can easily see that the optimal *h*-interval, where is the flat part in the Figure 1, can be taken as $[-1.5, -0.5]$. Figure 2 compares the 12-th order approximation solution obtained by HAM with the exact solution when *h* = -1. Moreover, applying Padé [6,6] approximation to the 12-th order approximate solution obtained by HAM yields HAPM solution, which are also displayed in Figure 2. Here, the difference between HAM and HAPM is negligible and are closer to the exact solution. On the other hand, Figure 3 gives the error between approximate solutions and the exact solutions. We would like to point out that convergence domain of approximate solutions increases via Padé approximant, see Figure 4, where *t* = 3 divergent for HAM but convergent for HAPM. Hence, HAPM, which couples the homotopy analysis method and the Padé approximation, is an efficient tool to obtain convergent solutions even for large values of *t*.

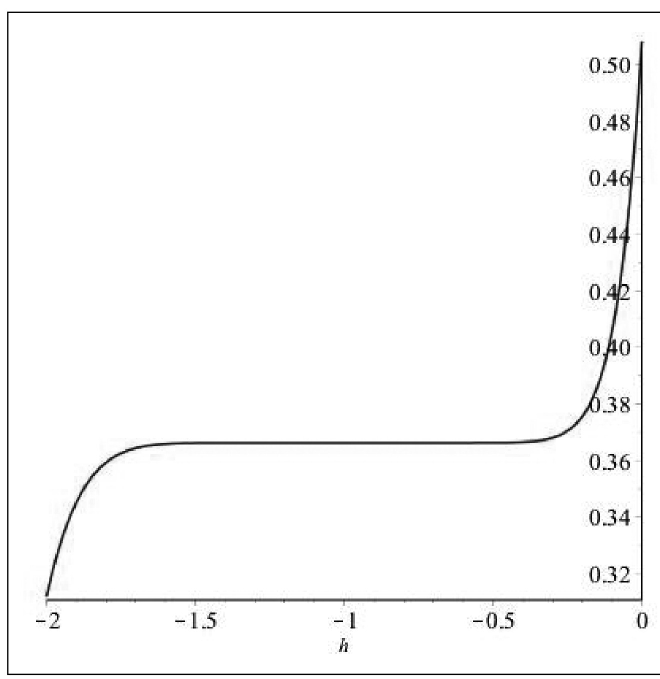

Figure 1. The *h*-graph to determine the optimum values of *h* for example 1.

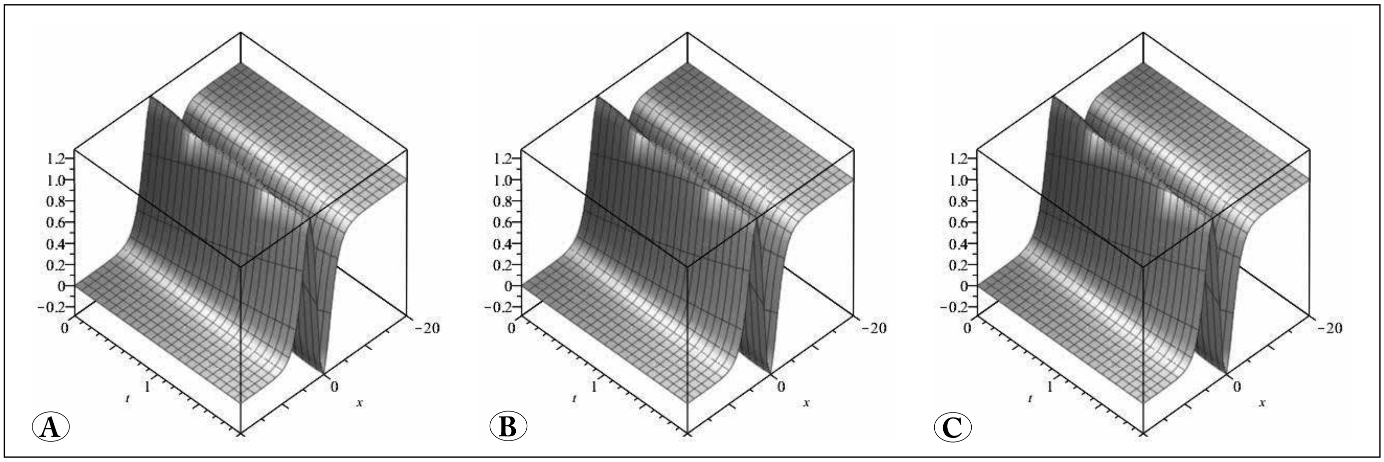

Figure 2. The exact solution of the third order dispersive Fisher equation is given by **(A)**, semi-analytical solutions obtained by HAM and HAPM of the third-order dispersive Fisher equation with (17) are given by **(B)** and **(C)**, respectively.

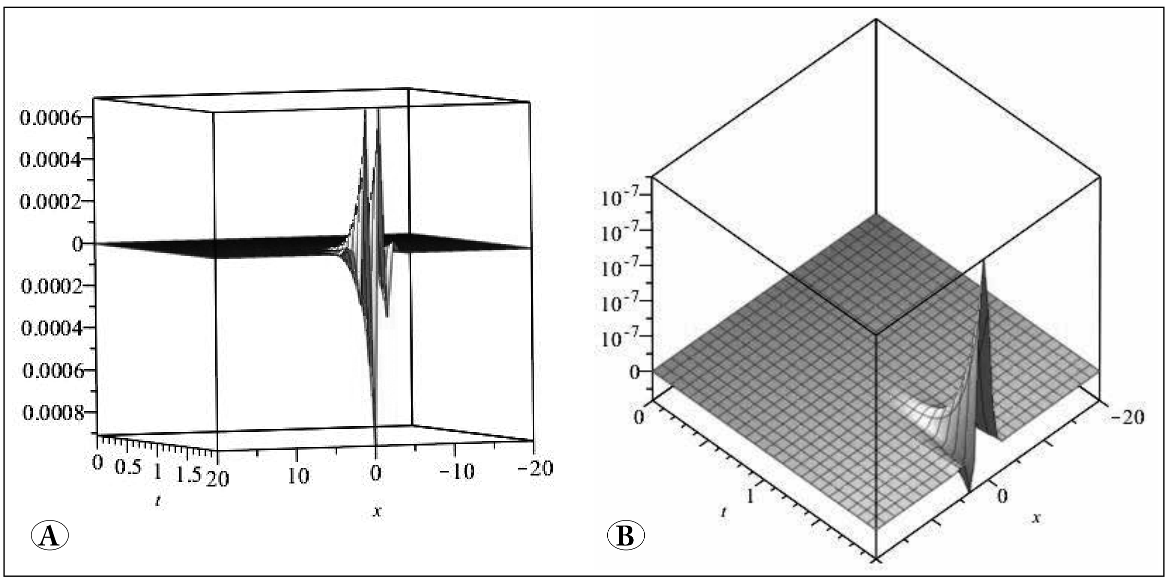

Figure 3. The error between the exact solution of the third-order dispersive Fisher equation and semi-analytical solutions obtained by HAM and HAPM with (17), **(A)** and **(B)**, respectively.

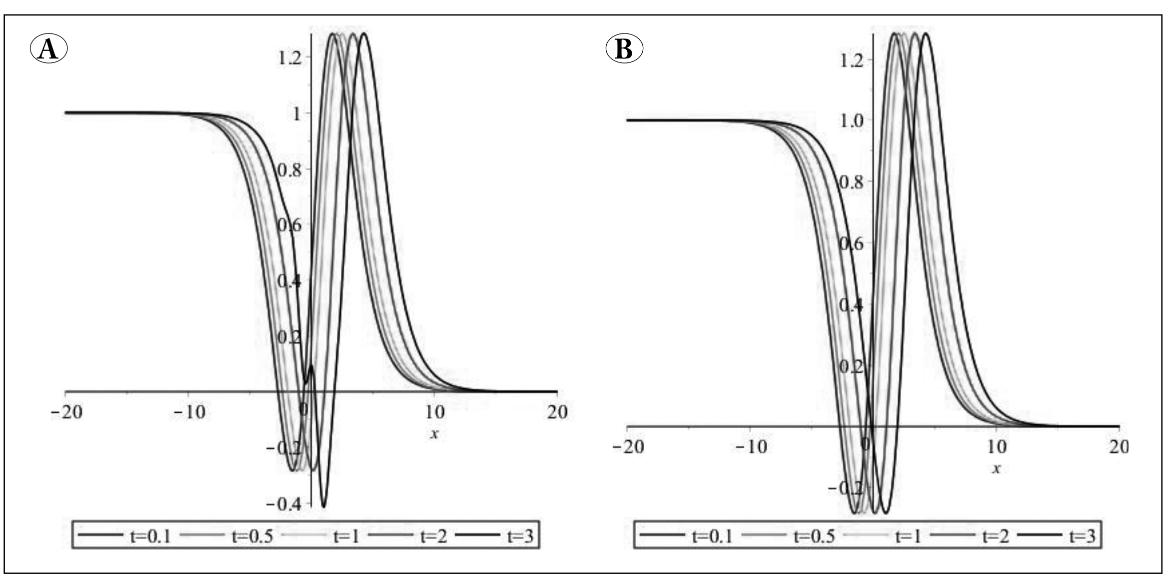

Figure 4: The effect of Padé Approximation to HAM with (17): **(A)** the approximate values of the solutions obtained by HAM, **(B)** the approximate values of the solutions obtained by HAPM.

Example 2: We now consider the equation (3) with

$$u(x,0)=\frac{1}{2}+\frac{3}{4}\tanh\frac{1}{60}30^{2/3}x-\frac{1}{4}\tanh\frac{1}{60}30^{2/3}x^{3}. \quad (18)$$

Similar to the previous example, using equation (16) with the initial condition (18), we have

$$u_1(x,t)=-\frac{19ht}{\cosh\frac{1}{60}30^{2/3}x^{4}}$$

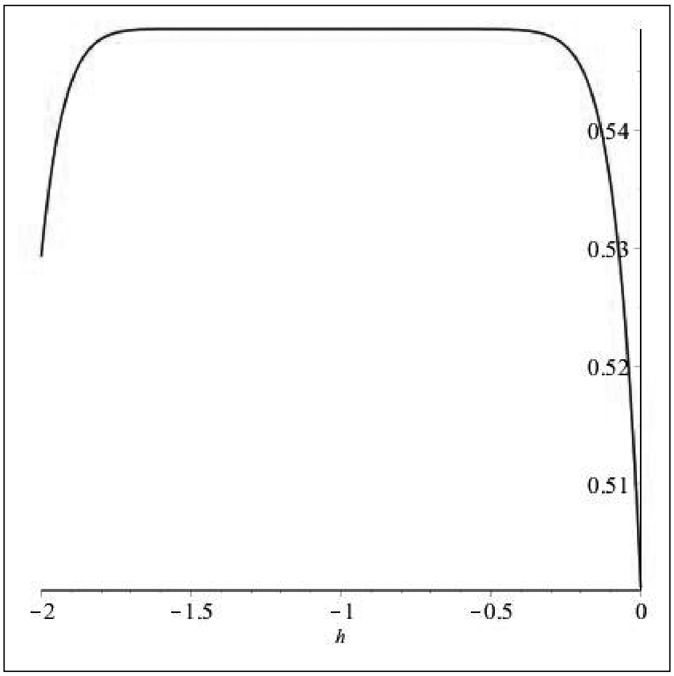

Figure 5. The h-graph to determine the optimum values of h for example 2.

$$u_2(x,t)=\frac{19}{2400\cosh\frac{1}{6}30^{2/3}x^{5}}ht\,30\cosh\frac{1}{60}30^{2/3}x+$$

$$30\cosh\frac{1}{60}30^{2/3}x\,h+19ht\sinh\frac{1}{60}30^{2/3}x$$

$$\vdots$$

and so on. Figure 5 displays the h-graph for the 12-th order approximate HAM solution, where h-interval can be seen as [-1.7,-0.4]. By taking h = -1, Figure 6 represents the 12-th order HAM solution and HAPM solution with Padé[6,6] approximation, for the different values of t.

4. Conclusion

In this study, the homotopy-based semi-analytical solutions, namely HAM and HAPM, are used to compare the performance for the solution of the third-order dispersive-Fisher equation. As underlined, the numerical performance of the methods shows parallelism with the theoretical perspective of third-order dispersive Fisher equation. The most important point is that convergence domain of approximate solutions increases via Padé approximant, where the solution is divergent for HAM but convergent for HAPM. Therefore the explanation of HAPM can be given as "HAPM couples the homotopy analysis method and the Padé approximation and it is an efficient tool to obtain convergent solutions even for large values of t. Illustrated figures, which are the application of Padé approximation to long time behaviour of HAM solution yields efficient results, support the proposed idea.

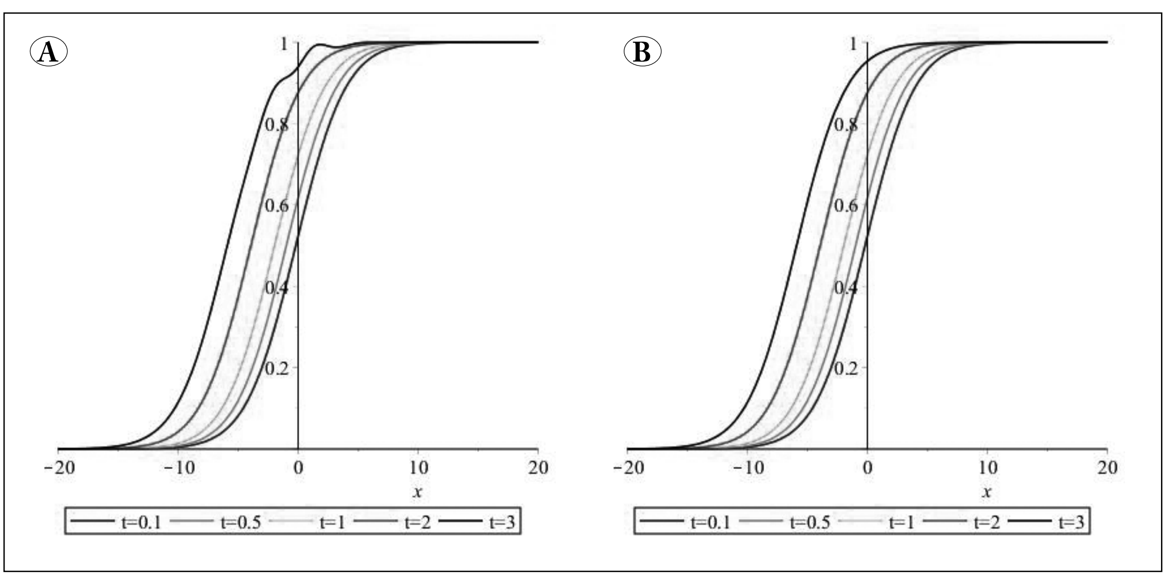

Figure 6. The effect of Padé Approximation to HAM with (18): **(A)** the approximate values of the solutions obtained by HAM, **(B)** the approximate values of the solutions obtained by HAPM.

5. References

Baker, GAJr., Graves-Morris, P. 1996. Padé Approximants. *Cambridge U.P.*

Brezenski, C. 1996. Extrapolation algorithms and Padé approximations. *Appl Numer Math*, 20 (3): 299–318

Fisher, RA. 1937. The wave of advance of advantageous genes. *Ann. Eugenics* 7:353-369.

He, JH. 1999. Homotopy perturbation technique. *Comput. Methods Appl. Mech. Engng.* 178 (3/4) : 257–262.

He, JH. 1998. An approximate solution technique depending upon an artificial parameter. *Commun. Nonlinear Sci. Numer. Simulat.* 3 (2): 92–97.

He, JH. 1998. Newton-like iteration method for solving algebraic equations. *Commun. Nonlinear Sci. Numer. Simulat.* 3 (2): 106–109.

He, JH. 2000. A coupling method of homotopy technique and perturbation technique for nonlinear problems. *Int. J. Nonlinear Mech*, 35 (1): 37–43.

He, JH. 2004. Comparison of homotopy perturbation method and homotopy analysis method. *Appl Math Comput* 156(2): 527-539.

Jegen, MD., Everett, ME., Schultz, A. 2001. Using homotopy to invert geophysical data. *Geophysics* 66 (6):1749–1760.

Kocak, H., Pinar, Z. 2017. On solutions of the fifth-order dispersive equations with porous medium type non-linearity. *Wave Random Complex Media*, DOI:10.1080/17455030.2017.1367438, 2017.

Kolmogorov, AN., Petrovskii, I. G., Piskunov, NS. 1937. Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem. *Bull. Moskov. Gos. Univ., Sect. A*, 1:1–26. English. transl. In: Dynamics of Curved Fronts, P. Pelce, Ed., Acad. Press, Inc., New York, 1988, 105–130.

Kumar, S., Singh, J., Kumar, D. , Kapoor, S. 2014. New homotopy analysis transform algorithm to solve volterra integral equation. *Ain Shams Eng J*, 5 (1): 243-246

Liao, SJ. 1992. The proposed homotopy analysis technique for the solution of nonlinear problems, *PhD thesis*, Shanghai Jiao Tong University.

Liao, SJ. 1999. An explicit, totally analytic approximation of Blasius' viscous flow problems. *Int. J. Non-Linear Mech.*, 34 (4): 759–778.

Liao, SJ. 2003. Beyond Perturbation: Introduction to the Homotopy Analysis Method. *Boca Raton: Chapman & Hall/CRC Press.*

Liao, SJ. 2005. Comparison between the homotopy analysis method and homotopy perturbation method. *Appl. Math. Compu.t*, 169 (2):1186-1194.

Liao, SJ. 2012. Homotopy Analysis Method in Nonlinear Differential Equations. *Berlin & Beijing: Springer & Higher Education Press.*

Liao, SJ.1992. A second-order approximate analytical solution of a simple pendulum by the process analysis method. *ASME J. Appl. Mech.* 59:970–975.

Marinca, V., Herişanu, N. 2008. Application of Optimal Homotopy Asymptotic Method for solving nonlinear equations arising in heat transfer. *Int. Commun. in Heat and Mass Transfer* 35 : 710–715.

Molabahramia, A., Khania, F. 2009. The homotopy analysis method to solve the Burgers–Huxley equation. *Nonlinear Analysis: Real World Applications* 10:589–600.

Pinar, Z., Kocak, H. 2018. Exact solutions for the third-order dispersive-Fisher equations. *Nonlinear Dyn*, 91(1), 421-426 .

Wazwaz, AM. 2008. Analytic study on Burgers, Fisher, Huxley equations and combined forms of these equations. *Appl Math Comp*, 195(2):754-761.