Journal of King Saud University – Science 32 (2020) 2190–2194

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Original article

Analytical study on the balancing principle for the nonlinear Klein–Gordon equation with a fractional power potential

Zehra Pinar

Tekirdağ Namık Kemal University, Faculty of Arts and Science, Department of Mathematics, 59030 Tekirdağ, Turkey

ARTICLE INFO

Article history:
Received 5 June 2018
Revised 21 February 2020
Accepted 23 February 2020
Available online 4 March 2020

Keywords:
Nonlinear Klein–Gordon Equation with a Fractional Power Potential
BBM equation
Exact solutions

ABSTRACT

In the literature, although many methods depend on the ansatz solution, there is not any specific rule to determine the degree of the ansatz i.e. balancing principle. The problem is especially seen with the fractional and the generalized nonlinear evolution equations. But the main thing is how many terms are needed to determine explicit solution of these type equations. As the previous work, in this work the balancing principle is generalized for large classes of nonlinear evolution equations, also includes fractional nonlinear models.

1. Introduction

Nonlinear models play key role in various filed of physics such as field theory, condensed matter physics, and hydrodynamics. (1 + 1)- and (2 + 1)-dimensional models can be seen more than (3 + 1)-dimensional models. Especially, the behavior of solutions of the sine-Gordon (Alfimov et al., 2000) and nonlinear Schrödinger (Richard et al., 2007) equations for spatially two and three-dimensional cases was numerically and analytically studied in detail by Ekomasov and Salimov (2014; 2015).

For the analytical solution of nonlinear models is tried to obtain via one of the computational methods such as the tanh method (Zhou et al., 2003; Ozis and Koroglu, 2008), sine–cosine method (Ma et al., 2010), G'/G-expansion method (Mirzazadeh et al., 2014; Ozis and Aslan, 2010), the Jacobi elliptic function expansion method (Yong et al., 2009), auxiliary equation method (Sirendaoreji, 2007; Lv et al., 2009; Yomba, 2008; Abdou, 2008; Lim et al., 2001; Pınar and Öziş, 2013a; 2013b), sub-equation method (Zhang, 2009; Lia et al., 2008; Yomba, 2006; 2005) etc.. The mentioned methods depending on the ansatz are chosen and the main problem is how many terms of the finite series i.e. ansatz is considered. Now, the general methodology for the mentioned methods is given as following:

For general, (1 + 1)-dimensional nonlinear model is considered

$$P(u, u_x, u_t, u_{xx}, u_{tx}, u_{tt}, \ldots) = 0 \tag{1}$$

and using an appropriate transformation Eq.(1) is reduced to Eq.(2)

$$Q(u, u_\zeta, u_{\zeta\zeta}, u_{\zeta\zeta\zeta} \ldots) = 0 \tag{2}$$

The analytical solutions of the Eq.(2) is considered as a finite series which is known ansatz

$$u(\zeta) = \sum_{i=0}^{N} a_i z^i(\zeta) \tag{3}$$

where $a_i, (i = 0, 1, \ldots, N)$ are constants and $z(\zeta)$ is the elementary function and changes respect to the considered method.

As it is seen that the main problem of these type methodologies is determining N and generally N is determined by a "balancing principle" and there is a classical determination but it cannot be applied all types of nonlinear models. Pınar and Öziş (2015a) proposed different determination for N and it is more general than exist one.

The value of N is always considered as a positive integer and to satisfy this condition, the variable transformations are done (Pınar and Öziş, 2015a). The review of all balancing principles is given by Pınar and Öziş (2015a) and they pointed that there is no unique balancing principle for every type nonlinear models.

Now the novel and generalized balancing principle is introduced in this work. It is suitable for all type of nonlinearities,

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

E-mail address: zpinar@nku.edu.tr

https://doi.org/10.1016/j.jksus.2020.02.032
1018-3647/

results as positive integer and also, the power is, in most cases, the least number that meets the final expansion (Eq. (3)). To obtain integer degree ansatz, the novel balancing principle (Eq. (4)) given by modular balance formula

$$N + n = qN + p(N + s),\ (\text{mod } n) \tag{4}$$

where the highest order linear term is $\frac{\partial^n u}{\partial x^n}$ and the highest order non-linear term is$u^q\left(\frac{\partial^s u}{\partial x^s}\right)^p$.

In the following section we exemplify some situations that do not work with the existing balancing principles either in the literature or in their work, Eq. (4) is lower than the anchor in which the power determined by the balancing principle proposed is. Therefore, the computation cost is also reduced via optimum degree of the ansatz.

In this work Lorentz-invariant model is solved via Bernoulli approximation method which has the given procedure and generalized BBM equation is solved via the extended auxiliary equation method.

2. The solution of Lorentz-invariant model

Nonlinear wave equations, especially most known are nonlinear Klein–Gordon equations, are used in several areas of physics and engineering such as hydrodynamics, condensed state physics, and field theory (Scott,2004; Braun et al., 2004; Knight et al., 2013; Dauxois and Peyrard, 2010; Saadatmand and Kurosh,2013; Sirendaoreji, 2007) and a quantified version of the relative energy–momentum relationship. Although (1 + 1) and (2 + 1) dimensional models (Knight et al., 2013; Saadatmand and Kurosh, 2013; Gonzalez et al., 2007; Efremidis et al., 2007; Fokas, 2006; Biswas et al., 2012) are the most studied, these equations can be easily generalized to higher dimensional space, e.g., for spherical symmetry.

Now one of the most important models of physics is a Lorentz-invariant model that has solutions in the form of plane waves and is a modification of the previously considered equations (Ekomasov and Salimov, 2014):

$$u_{rr} + 2\frac{u_r}{r} - u_{tt} = u^{m/n}\left({u_r}^2 - {u_t}^2\right)^{k/s} \tag{5}$$

where m, n, s are odd natural numbers and k is even natural number. These types of equations are drawn attention for plane waves. At this point, the analytical solutions are important to obtain time periodic solutions in a large number of models and various dimensions of space time.

When $m = 3, k = 2, n = s = 13$, Eq. (5) is rewritten

$$u_{rr} + 2\frac{u_r}{r} - u_{tt} = u^{3/13}\left({u_r}^2 - {u_t}^2\right)^{2/13} \tag{6}$$

with the wave transformation $\zeta = \mu r + ct$, $c \neq 0$ and $\mu \neq 0$, as a result the reduced equation is

$$(\mu^2 - c^2)u'' + \frac{2\mu}{r}u' = (\mu^2 - c^2)^{2/13}(u')^{4/13}u^{3/13} \tag{7}$$

and using classical balancing principle $N = -\frac{11}{3}$ is obtained, but how the finite series can be determined with this upper bound?

For Eq. (7), reduced form of Eq. (6), with the novel proposed balancing principle (i.e. Eq.(4)) $N \equiv 0 \equiv 2(\ mod\ 2)$ is obtained.

Hence, the solution of the Eq. (7) can be given by

$$u(\zeta) = \sum_{i=0}^{N=2} a_i z^i(\zeta) = a_0 + a_1 z(\zeta) + a_2 z^2(\zeta),\ \zeta = \mu r + ct$$

where $z(\zeta)$is considered as a solution of the variable coefficient Bernoulli differential equation

$$\frac{dz(\xi)}{d\xi} = P(\xi)z(\xi) + Q(\xi)z^n(\xi) \tag{8}$$

where $P(\zeta)$ and $Q(\xi)$ are any functions and $n > 1$ is an integer. New precise analytical solutions of the Bernoulli equation (i.e. Eq. (8)) have been given by Pınar and Öziş (2015b) and Pinar and Kocak (2018) for different coefficient functions and $n = 2$under twenty different conditions.

For $n = 2$, as a result of nonlinear algebraic system, the parameters are obtained

$$P(\zeta) = \frac{2\mu}{r(c^2 - \mu^2)}, Q(\zeta) = -\frac{\exp\left(-\frac{4\zeta\mu}{r(c^2-\mu^2)}\right)\left(2a_2\mu\exp\left(\frac{4\zeta\mu}{r(c^2-\mu^2)}\right) + a_1 r C_1(\mu^2 - c^2)\right)}{r(c^2 - \mu^2)a_1}, a_2 = \frac{a_1 r C_1(c^2 - \mu^2)\sqrt{3}}{6\exp\left(\frac{4\zeta}{r(c^2-\mu^2)}\right)},$$

Hence the solution of the Bernoulli differential equation is

$$z(\zeta) = 2(\mu^2 - 5\mu + 6)a_1\exp\left(\frac{2\zeta}{r(c^2 - \mu^2)}\right) \Big/ \left(\begin{array}{c} c^2 a_1 r C_1 \exp\left(\frac{2(\mu-2)\zeta}{r(c^2-\mu^2)}\right)(\mu - 3 + \mu^3 - 3\mu^2) \\ +2a_1C_2(6 - 5\mu + \mu^2) + 4a_2\exp\left(\frac{\zeta}{r(c^2-\mu^2)}\right)^2(\mu^2 - 4\mu + 4) \end{array}\right)$$

The solution is given for large times by Fig. 1 and behavior of the amplitude in the center of the pulson for Eq. (6) at small times and large times are seen in Fig. 2.

To be sure, also we need to check that the stabilization of the oscillation by Eq.(6)

$$c(t) = \frac{1}{t}\int_0^1 |u(0, t)|dt \tag{9}$$

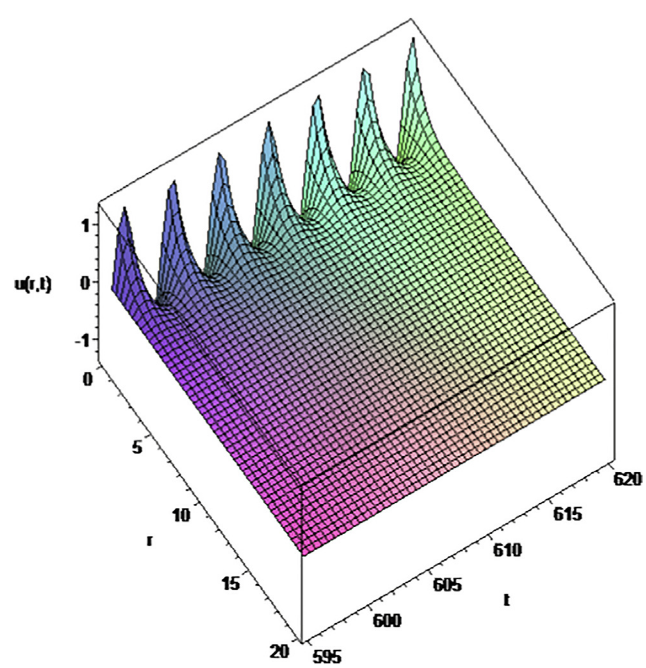

Fig. 1. The solution of the Eq.(6) for $a_0 = 2e^{-r}\cos(\frac{t\pi}{2}), a_1 = 0.0001t, C_1 = 2, C_2 = 1, \mu = 1, c = -2$

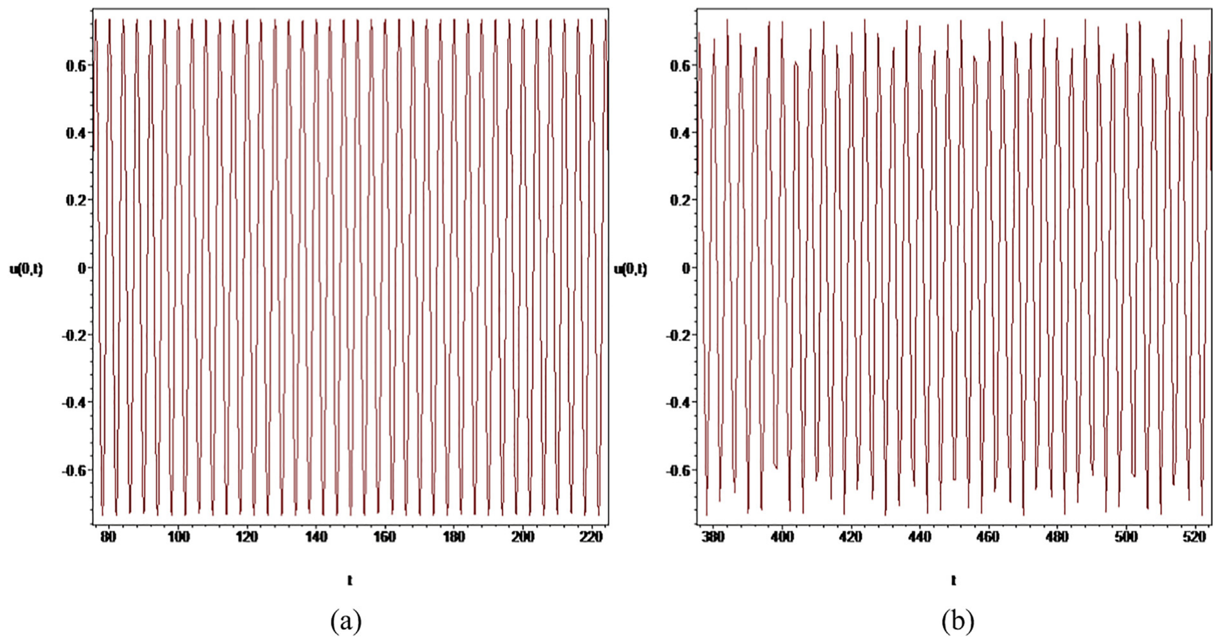

Fig. 2. Behavior of the amplitude in the center of the pulson for Eq. (6) at small times and large times, respectively.

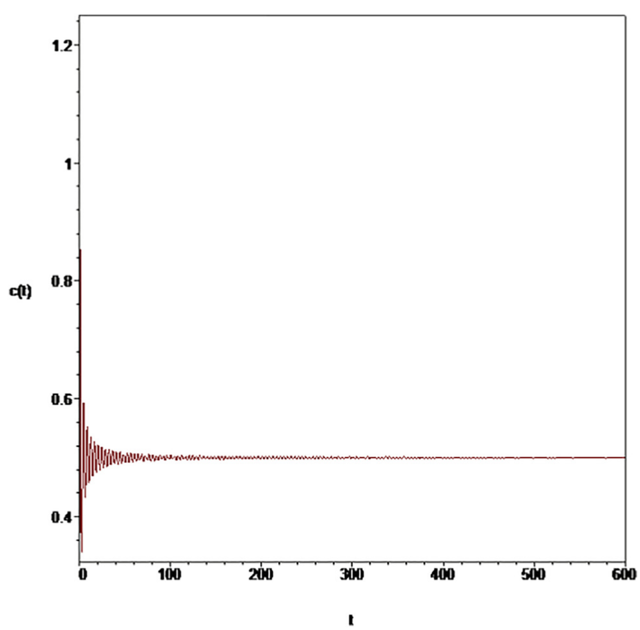

Fig. 3. Time evolution of the quantity $c(t)$for Eq. (6).

The plot of the time evolution of the quantity $c(t)$for Eq. (6) is given by Fig. 3.

For the following example, in (Pınar and Ozis, 2013b), ansatz is taken first degree, in (Layeni and Akinola, 2010) ansatz is $2/n$-th degree and in (Wazwaz and Helal, 2005) the degree of ansatz is $\frac{2}{n-1}$-th for the general Benjamin–Bona–Mahony (BBM) equation. In (Wazwaz and Helal, 2005), it is said that degree of ansatz should be integer. This means that when $n = 3$, the result is $N = 1$ and in the similar manner if $n = 2$, $N = 2$is obtained. For general equations, appropriate n is chosen so that N is an integer. So, we do not have the general equation solution. By Pınar and Öziş (2015a), ansatz with fractional degree is used as a result new solutions for the general equation is obtained. Hence, to obtain ansatz with integer degree, the novel balancing principle (Eq. (4)) given by modular balance formula is considered.

3. The generalized Benjamin–Bona–Mahony (BBM) equation

The generalized BBM equation (Nickel, 2007; Wang et al., 2014), which is an improvement of the Korteweg–de Vries equation (KdV equation) for modeling long surface gravity waves of small amplitude, is considered;

$$u_t + au^n u_x + u_x + u_{xxx} = 0. \tag{10}$$

with the wave transformation $\zeta = \mu x + ct$, $c \neq 0$ and $\mu \neq 0$, as a result the reduced equation

$$cu' + \mu u^n u' + \mu u' + \mu^3 u''' = 0 \tag{11}$$

For Eq. (11), we obtain zero-degree ansatz by which the trivial solution of Eq.(11) is hold. In addition, Eq. (11) is an integrable equation. If Eq. (11) is integrated respect to ζ, then we obtain

$$cu + \frac{\mu u^{n+1}}{n+1} + \mu u + \mu^3 u'' = 0. \tag{12}$$

Using the novel balancing principle (Eq.(4)), $2/n$- degree ansatz for Eq. (12) is obtained. For $2/n$- degree ansatz, we have several cases as following.

Case 1. *Considering $n = 1$, the ansatz is second degree and Eq. (12) become*

$$cu + \frac{\mu u^2}{2} + \mu u + \mu^3 u'' = 0. \tag{13}$$

We get solution of Eq. (13) using the second degree ansatz and the extended auxiliary equation $(z'(\zeta))^2 = a_2 z^2(\zeta) + a_6 z^6(\zeta)$, the solution is plotted in Fig. 4.

Case 2. *Considering $n = 2$, the ansatz is first-degree and Eq. (12) becomes*

$$cu + \frac{\mu u^3}{3} + \mu u + \mu^3 u'' = 0. \tag{14}$$

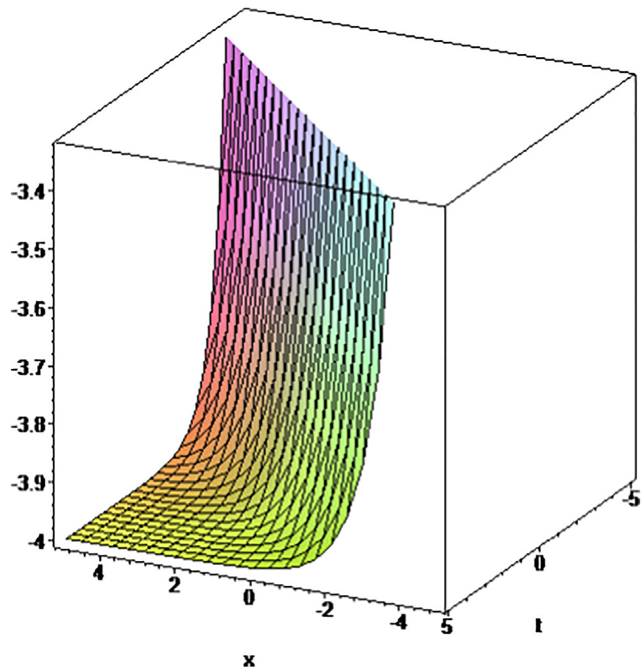

Fig. 4. This solution of Case 1 is obtained using second degree ansatz.

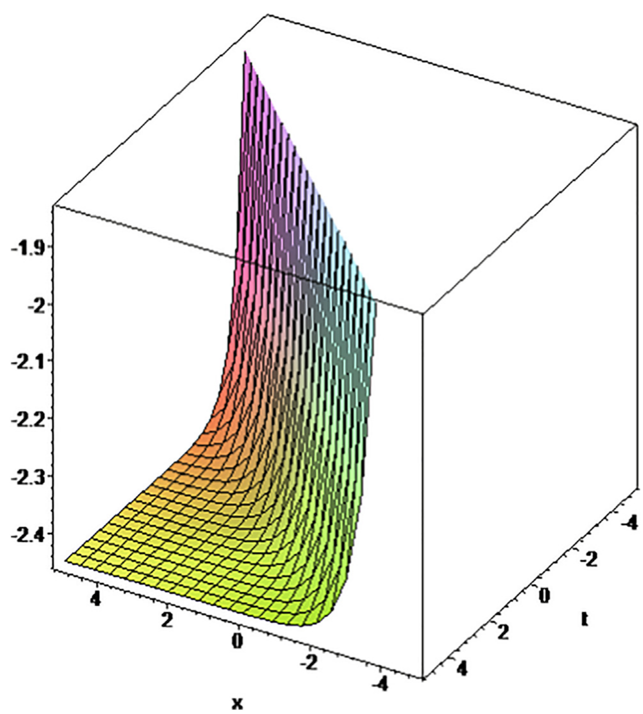

Fig. 5. This solution of Case 2 is obtained using first degree ansatz.

We get solution of Eq. (14) using the first degree ansatz and the extended auxiliary equation $(z'(\zeta))^2 = a_2z^2(\zeta) + a_6z^6(\zeta)$, the solution is plotted in Fig. 5.

Now, we compare same n value with different ansatzs. If we take for $n = 1$, first and second degree ansatzes with the extended auxiliary equation $(z'(\zeta))^2 = a_2z^2(\zeta) + a_4z^4(\zeta) + a_6z^6(\zeta)$

Here, we have important question, if we take non-integer degree ansatz, what kind of solutions are obtained. For this example, if we take $n = 3$, ansatz is 2/3-degree but it is equivalence to 2 with respect to $(mod2)$. Using the extended auxiliary equation $(z'(\zeta))^2 = a_6z^6(\zeta)$ and $u(\zeta) = a_0 + a_1z^{2/3}(\zeta)$ ansatz, we obtain

$$u(x,t) = a_0 + \frac{a_1}{\left(2\mu(x + (-\mu - \frac{5}{8}a\mu a_0{}^3)t) + c_1\right)^{1/3}} \tag{15}$$

The solutions which are obtained using second degree ansatz and 2/3-degree ansatz have same behaviors.

As mentioned above, the value of N is always considered as a positive integer and to satisfy this condition, the variable transformations or if the equation is integrable, the equation is integrated considering the integration constant is zero, etc.. Instead of all these reductions, with the novel principle the problem is solved.

4. Conclusion

The common balancing principles (i.e. determines the power N of ansatz (cf. Eq.(4)) usually by balancing the highest order linear term in the equation with the highest order nonlinear term) work only for positive integer values. In this study, a prosperous balancing principle has been proposed that works for the positive integer one and with least power of the ansatz. To verify our objective well-known problems in the literature is given which works parallel to our goal. In addition, the proposed balancing principle make possible for finding new travelling wave solutions or analytical solutions of the nonlinear problems due to introducing a novel ansatz(s) as in the given examples. The future work will focus on symmetries of the Lorentz-invariant model.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Abdou, M.A., 2008. A generalized auxiliary equation method and its applications. Nonlinear Dyn 52, 95–102.
Alfimov, G.L., Evans, W.A.B., Vazquez, L., 2000. On radial sine-Gordon breathers. Nonlinearity 13, 1657.
Biswas, A., Khalique, C.M., Adem, A.R., 2012. Traveling wave solutions of the nonlinear dispersive klein-gordon equations. J. King Saud Univ. Sci. 24 (4), 339–342.
Braun, O.M., Yu, S., Kivshar, 2004. The Frenkel-Kontorova model: concepts, methods, and applications. Springer, Berlin.
Dauxois, T., Peyrard, M., 2010. Physics of solitons. Cambridge Univ. Press, New York.
Efremidis, N.K., Hizanidis, K., Malomed, B.A., Trapani, P.D., 2007. Three-dimensional vortex solitons in self-defocusing media. Phys. Rev. Lett. 98, 113901.
Ekomasov, E.G., Salimov, R.K., 2014. On localized long-lived three-dimensional solutions of the nonlinear Klein-Gordon equation with a fractional power potential. JETP Lett. 100, 477.
Ekomasov, E.G., Salimov, R.K., 2015. On the nonlinear (3 + 1)-dimensional Klein-Gordon equation allowing oscillating localized solutions. JETP Lett. 102, 122.
Fokas, A.S., 2006. Integrable nonlinear evolution PDEs in 4+2 and 3+1 dimensions. Phys. Rev. Lett. 96, 190201.
Gonzalez, J.A., Bellorin, A., Guerrero, L.E., 2007. Kink-soliton explosions in generalized Klein-Gordon equations. Solitons Fractals 33, 143.
Knight, C.J.K., Derks, G., Doelman, A., Susanto, H., 2013. Stability of stationary fronts in a non-linear wave equation with spatial inhomogeneity. J. Diff. Eq. 254, 408.
Layeni, O.P., Akinola, A.P., 2010. A new hyperbolic auxiliary function method and exact solutions of the mBBM equation. Commun. Nonlinear Sci. Numer. Simulat. 15, 135–138.
Lia, B., Chena, Y., Q Lia, Y., 2008. A Generalized sub-equation expansion method and some analytical solutions to the inhomogeneous higher-order nonlinear Schrödinger equation. Z Naturforsch 63a, 763–77.
Lim, C.W., Wu, B.S., He, Lh., 2001. A new approximate analytical approach for dispersion relation of the nonlinear Klein-Gordon equation. Chaos 11 (4), 843–848.
Lv, X., Lai, S., Wu, Y.H., 2009. An auxiliary equation technique and exact solutions for a nonlinear Klein-Gordon equation. Chaos Solitons Fract 41, 82–90.
Ma, W.X., Huang, T., Zhang, Y.A., 2010. Multiple exp-function method for nonlinear differential equations and its applications. Phys Scr 82, 065003.

Mirzazadeh, M., Eslami, M., Biswas, A., 2014. Soliton Solutions of the generalized klein-gordon equation By G'/G-expansion method. Comput. Appl. Mathemat. 33 (3), 831–839.

Nickel, J., 2007. Elliptic solutions to a generalized BBM equation. Physics Letters A 364, 221–226.

Ozis, T., Aslan, I., 2010. Application of the G'/G-expansion method to Kawahara type equations using symbolic computation. Appl Math Comput 216, 2360–2365.

Ozis, T., Koroglu, C., 2008. A novel approach for solving the Fisher equation using exp-function method. Phys Lett A 372, 3836–3840.

Pinar, Z., Kocak, H., 2018. Exact solutions for the third-order dispersive-Fisher equations. Nonlinear Dyn 91, 421–426.

Pınar, Z., Öziş, T., 2013. The Periodic Solutions to Kawahara Equation by means of the auxiliary equation with a sixth-degree nonlinear term. J. Mathemat., (Doi: 10.1155/2013/106349).

Pınar, Z., Ozis, T., 2013b. An observation on the periodic solutions to nonlinear physical models by means of the auxiliary equation with a sixth-degree nonlinear term. Commun. Nonlinear Sci. Numer. Simulat. 18, 2177–2187.

Pınar, Z., Öziş, T., 2015a. Observations on the class of "Balancing Principle" for nonlinear PDEs that can be treated by the auxiliary equation method. Nonlinear Anal. Real World Appl. 23, 9–16.

Pınar, Z., Öziş, T., 2015. A remark on a variable-coefficient Bernoulli equation based on auxiliary equation method for nonlinear physical systems, arXiv:1511.02154.

Richard, S., Tasgal, Y.B., Malomed, B.A., 2007. Optoacoustic solitons in bragg gratings. Phys. Rev. Lett. 98, 243902.

Saadatmand, D., Kurosh, J., 2013. Collective-coordinate analysis of inhomogeneous nonlinear klein-gordon field theory. Braz. J. Phys. 56, 43.

Scott, A., 2004. Encyclopedia of Nonlinear Science. Routledge, New York.

Sirendaoreji, S., 2007. Auxiliary equation method and new solutions of Klein-Gordon equations. Chaos Solitons Fract. 31, 943–950.

Wang, G.W., Xu, T.X., Abazari, R., Jovanoski, Z., Biswas, A., 2014. Shock waves and other solutions to the benjaminbona-mahoney-burgers equation with dual-power law nonlinearity. Acta Physica Polonica A 126 (6), 1221–1225.

Wazwaz, A.M., Helal, M.A., 2005. Nonlinear variants of the BBM equation with compact and noncompact physical structures. Chaos Solitons Fractals 26, 767–776.

Yomba, E., 2005. The extended Fan's sub-equation method and its application to KdV-MKdV, BKK and variant Boussinesq equations. Phys. Lett. A 336, 463–476.

Yomba, E., 2006. The modified extended fan sub-equation method and its application to the (2+1)-dimensional Broer–Kaup–Kupershmidt equation. Chaos Solitons Fract 27, 187–196.

Yomba, E., 2008. A generalized auxiliary equation method and its application to nonlinear Klein-Gordon and generalized nonlinear Camassa-Holm equations. Phys. Lett. A 372, 1048–1060.

Yong, X., Zeng, X., Zhang, Z., Chen, Y., 2009. Symbolic computation of Jacobi elliptic function solutions to nonlinear differential-difference equations. Comput. Math. Appl. 57, 1107–1114.

Zhang, H., 2009. A note on some sub-equation methods and new types of exact travelling wave solutions for two nonlinear partial differential equations. Acta Appl. Math. 106, 241–249.

Zhou, Y., Wang, M., Wang, Y., 2003. Periodic wave solutions to coupled KdV equations with variable coefficient. Phys. Lett. A 308, 31–36.