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Polynomial Affine Translation Surfaces in Euclidean 3-Space

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ABSTRACT: In this paper we study the polynomial affine translation surfaces in E^3 with constant curvature. We derive some non-existence results for such surfaces. Several examples are also given by figures.

Key Words: Affine translation surface, polynomial translation surface,Gaussian curvature, mean curvature.

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1. Introduction

A surface in Euclidean 3-space is called a *translation surface* if it is the graph surface of the function

$$z(x,y) = f(x) + g(y),$$

where f and g are smooth functions. Such surfaces are obtained by translating two planar curves. This class of the surfaces are well-studied classical surfaces in Euclidean and Lorentzian space [1,2,3,6,9].

A *polynomial translation surface* [8,10] is parametrized by

$$r : U \subseteq E^2 \to E^3, (x,y) \mapsto r(x,y) = (x, y, f(x) + g(y)),$$

where f and g are polynomial functions on U.

Most recently H. Liu and Y. Yu introduced a new translation surfaces so-called affine translation surfaces. The *affine translation surface* in Euclidean 3-space is defined as a parameter surface $r(u,v)$ in E^3 which can be written as

$$r(u,v) = (u, v, f(u) + g(v + au)) \ ,$$

for some non zero constant a and smooth functions $f(u)$ and $g(v + au)$.

The authors classified minimal affine translation surfaces in three dimensional Euclidean space. M. Magid and L. Vrancken [7] considered affine translation surface

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with constant sectional curvature in 4-dimensional affine space, by proving that such surfaces must be flat and one of the defining curves must be planar. Affine translation surfaces with constant Gaussian curvature in 3-dimensional affine space are investigated by Y. Fu and Z. Hou and they obtained a complete classification of such surfaces [4]. Also, Y. Yuan and H. L. Liu dealth with translation surfaces of some new types in 3-Minkowski space [11].

In this paper we investigate the affine translation surfaces with constant curvature in E^3, then we provided non-existence theorems for these surfaces.

2. Polynomial Affine Translation Surface with Constant Gaussian and Mean Curvature

Let $\langle\, , \rangle$ denote the standart scalar product on E^3 and let $\|\,.\,\|$ be the induced norm. Consider the affine translation surface M in E^3 parametrized by

$$r : U \subseteq E^2 \to E^3, (x, y) \mapsto r(x, y) = (x, y, f(x) + g(y + ax)), \tag{2.1}$$

where f and g are real-valued and smooth functions on U and a is a non-zero constant. Then the first fundamental form of M can be written as

$$I = Edx^2 + 2Fdxdy + Gdy^2,$$

where

$$\begin{aligned} E &= \langle r_x, r_x\rangle = 1 + (f' + ag')^2, \\ F &= \langle r_x, r_y\rangle = g'(f' + ag'), \\ G &= \langle r_y, r_y\rangle = 1 + g'^2, \end{aligned}$$

and here $f' = \dfrac{df(x)}{dx}$ and $g' = \dfrac{dg(v)}{dv} = \dfrac{dg(y+ax)}{d(y+ax)}$ for $v = y + ax$. The unit normal vector field so-called the Gauss map of M is given by

$$N = \frac{r_x \times r_y}{\|r_x \times r_y\|} = \frac{(-(f' + ag'), -g', 1)}{\sqrt{1 + (f' + ag')^2 + g'^2}}.$$

Then the second fundamental form of M is

$$II = Ldx^2 + 2Mdxdy + Ndy^2,$$

where

$$\begin{aligned} L &= \langle r_{xx}, N\rangle = \left(f''^2 g''\right) D^{-1}, \\ M &= \langle r_{xy}, N\rangle = ag'' D^{-1}, \\ N &= \langle r_{yy}, N\rangle = g'' D^{-1}, \end{aligned}$$

and here $D^2 = EG - F^2 = 1 + (f' + ag')^2 + g'^2$. Hence the Gauss and mean curvatures of M are given, respectively,

$$K = \frac{LN - M^2}{EG - F^2} = f''g''D^{-4} \tag{2.2}$$

and

$$H = \frac{LG - 2FM + NE}{2\left(EG - F^2\right)} = \frac{1}{2}\left[f''\left(1 + g'^2\right) + g''\left(1 + a^2 + f'^2\right)\right] D^{-3}. \tag{2.3}$$

Note that the affine translation surface given by (2.1) is flat, i.e. $K \equiv 0$, if and only if at least one of f or g is a linear function.

Example 2.1. *Let M be an affine translation surface in E^3 parametrized by*

$$r\left(x, y\right) = \left(x, y, x^2 + x + y\right), \left(x \in \left(-2, 1\right), \, y \in \left(-2, 1\right)\right).$$

It is easy to see that M is a parabolic cylinder and flat. It can be plotted as in Fig.1.

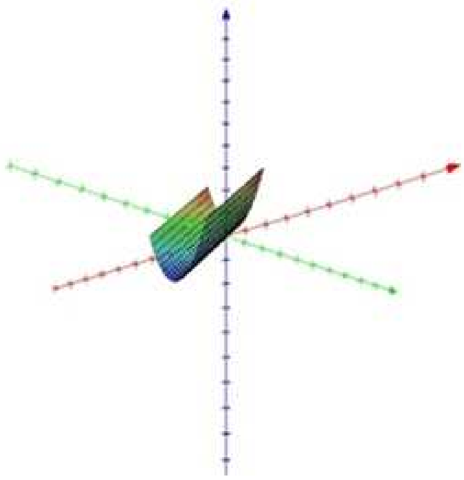

Figure 1:

H. Liu and Y. Yu [5] proved the classification theorem for minimal affine translation surfaces in the following

Theorem 2.1. *Let $r\left(x, y\right) = \left(x, y, z\left(x, y\right)\right)$ be a minimal affine translation surface. Then either $z\left(x, y\right)$ is linear or can be written as*

$$z\left(x, y\right) = \frac{1}{c} \log \frac{\cos\left(c\sqrt{1 + a^2} x\right)}{\cos\left[c\left(y + ax\right)\right]}. \tag{2.4}$$

The minimal translation surface given by (2.4) is called *generalized Sherk surface* or *affine Sherk surface* in E^3.

Example 2.2. *Let M be an affine Sherk surface in E^3 given by*

$$r(x,y)=\left(x,y,2\ln\cos\left(\frac{\sqrt{2}}{2}x\right)-2\ln\cos\left(\frac{1}{2}y+\frac{1}{2}x\right)\right),(x\in(-2,2)\,,\,y\in(-2,2))$$

We plot it as in Fig.2.

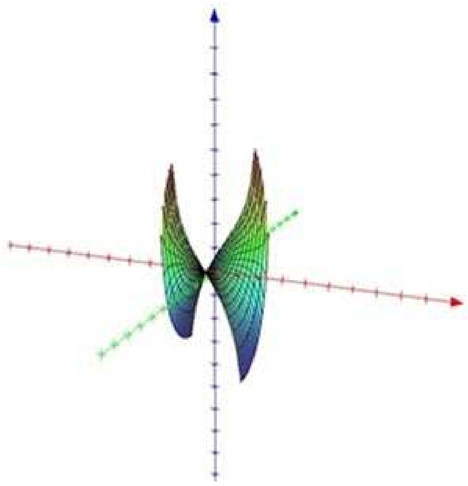

Figure 2:

Now, we consider the polynomial affine translation surfaces parametrized by

$$r:U\subseteq E^2\to E^3,(x,y)\mapsto r(x,y)=(x,y,f(x)+g(y+ax)),$$

where f and g are polynomial functions on U. Therefore, the following non-existence results for polynomial affine translation surfaces can be provided.

Theorem 2.2. *There does not exist a polynomial affine translation surface with non-zero constant Gaussian curvature in E^3.*

Proof: Let M be a polynomial affine translation surface with constant Gaussian curvature. From (2.2) we have $f''g''\neq 0$. Differentiating (2.2) with respect to y, we get

$$g'''\left(1+(f'+ag')^2+g'^2\right)-4g''^2\left(a\left(f'+ag'\right)+g'\right)=0, \tag{2.5}$$

Denoting f' and g' by α and β, respectively, we obtain

$$\beta''\left(1+(\alpha+a\beta)^2+\beta^2\right)-4\beta'^2\left(a\left(\alpha+a\beta\right)+\beta\right)=0. \tag{2.6}$$

Suppose that the polynomials α and β are given by

$$\alpha = b_m u^m + b_{m-1} u^{m-1} + ... + b_1 u + b_0$$

and

$$\beta = c_n v^n + c_{n-1} v^{n-1} + ... + c_1 v + c_0$$

where b_m and c_n are non-zero constants. Replacing α and β in (2.6) we get a polynomial expression in u and v vanishing identically, i.e., all the coefficients are zero. Let us consider some cases of equation (2.6)

Case 1. $m, n \geq 2$

i. Suppose that $m > n\,(\geq 2)$. The dominant term according to $u^{2m}v^{n-2}$ which comes from $\beta'' + \beta''\alpha^2 + 2a\alpha\beta\beta''$ having the coefficient $b_m^2 c_n n\,(n-1)$. This cannot vanish since $b_m, c_n \neq 0$ and $m > n \geq 2$.

ii. Suppose that $n > m\,(\geq 2)$ Using similar way, this case cannot occur.

ii. Suppose that $m = n\,(\geq 2)$ This case can be treated in similar way.

Case 2. $m, n \geq 1$

i. $m > n = 1$. We get $\beta = cv + d$ with real constants c, d and $c \neq 0$. If we consider this situation in equation (6), the coefficient of highest degree u^m comes from $-4a\alpha\beta'^2 + (-4 - 4a^2)\beta\beta'^2$ having the coefficient $-4ab_m c^2$. Then this expression cannot occur since $b_m, c \neq 0$.

ii. $n > m = 1$. From the similar way, this case cannot occur.

Case 3. $m \geq n = 0$(or $n \geq m = 0$) this situation is not possible since $f''g'' \neq 0$.

So, in every case, we obtain that there is no a polynomial affine translation surfaces with constant Gaussian curvature. □

So, the following result can be given

Corollary 2.3. *If the Gaussian curvature of a polynomial affine translation surfaces in E^3 is equal to a constant, the constant must be zero.*

Theorem 2.4. *There does not exist a polynomial affine translation surface with constant mean curvature in E^3 .*

Proof: Suppose that M is a polynomial affine translation surface with constant mean curvature. Differentiating equation (2.3) with respect to y, we get

$$\left[2f''g'g'' + g'''\left(1 + a^2 + f'^2\right)\right] - \tfrac{3}{2}\left(f''\left(1 + g'^2\right) + g''\left(1 + a^2 + f'^2\right)\right)$$
$$\left(2\left(f' + ag'\right)ag'' + 2g'g''\right)\left(1 + \left(f' + ag'\right)^2 + g'^2\right)^{-1} = 0$$

Denoting f' by α and g' by β we have

$$\begin{aligned}&\left[2\alpha'\beta\beta'+\beta''\left(1+a^2+\alpha^2\right)\right]\left[1+(\alpha+a\beta)^2+\beta^2\right]\\&-3\left(\alpha'\left(1+\beta^2\right)+\beta'\left(1+a^2+\alpha^2\right)\right)\left((\alpha+a\beta)\,a\beta'+\beta\beta'\right)=0\end{aligned} \tag{2.7}$$

Let us assume α and β are polynomials given by

$$\alpha=b_m u^m+b_{m-1}u^{m-1}+...+b_1u+b_0$$

and

$$\beta=c_n v^n+c_{n-1}v^{n-1}+...+c_1v+c_0$$

where b_m and c_n are non-zero constants. Substiuting α and β in (2.7) we get a polynomial expression in u and v vanishing identically that is all the coefficients are zero.

Let us consider some cases of equation (2.7) :

Case 1. $m, n \geq 2$

i. Suppose that $m > n\,(\geq 2)$. The dominant term according to $u^{4m}v^{n-1}$ which comes from $a^2\beta''\alpha^2+\beta''\alpha^4+2a\alpha\beta\beta''$ having the coefficient $b_m^4 c_n n\,(n-1)$. This cannot vanish since $b_m, c_n \neq 0$and $m > n \geq 2$.

ii. Suppose that $n > m\,(\geq 2)$ and $m = n\,(\geq 2)$. It is easy to see that these cases can be treated using similar method mentioned above.

Case 2. $m, n \geq 1$

i. $n > m = 1$. We get $\alpha = bu + d$ with real constants b, d and $b \neq 0$. If we consider this situation in equation (2.7), the coefficient of highest degree v^{4n-1} comes from $4\alpha'\beta^3\beta' - 3a^2\alpha'\beta^3\beta' - 3\alpha'\beta^3\beta'$ having the coefficient$\left(1-3a^2\right)nbc_n^4$. Then this case cannot occur since $b, c_n \neq 0$.

ii. $m > n = 1$. Using similar way, this case cannot occur.

Case 3. $m, n \geq 0$

i. $m \geq n = 0$. Then β is a constant, so the equation (2.7) is satisfied. But if β is constant from the equation (2.3) α is not be a polynomial. It is a contradiction, so this situation cannot occur.

ii. $n \geq m = 0$. Then α ($\alpha = b$) is constant, so the equation (2.7) can rewrite with this case in the following way

$$\begin{aligned}&\left[\beta''\left(1+a^2+b^2\right)\right]\left[1+(b+a\beta)^2+\beta^2\right]\\&-3\left(\beta'\left(1+a^2+b^2\right)\right)\left((b+a\beta)\,a\beta'+\beta\beta'\right)=0\end{aligned}$$

Using the same idea like in case 1,2 we can say that this situation cannot occur since $b_m \neq 0$.

So, the proof is completed. □

Then the following result is given.

Corollary 2.5. *If the mean curvature of a polynomial affine translation surfaces in E^3 is equal to a constant, the constant must be zero.*

Example 2.3. *Let M be a polynomial affine translation surface in E^3 parametrized by*

$$r\left(x, y, x^4 + (x+y)^2 - \mathrm{x} - \mathrm{y}\right), (x \in (-1,1), \; y \in (-1,1))$$

It can be plotted as in Fig.3.

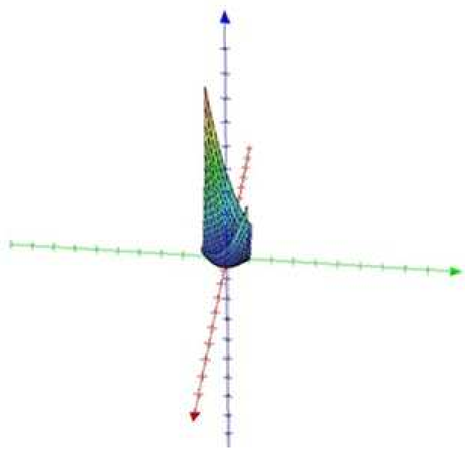

Figure 3:

3. A Further Application

As a further application we can choose the functions α and β as the exponential ones, i.e., $\alpha = c_1 e^u$ and $\beta = c_2 e^v$ where c_1 and c_2 are real numbers and $c_1, c_2 \neq 0$. Then the equation (2.6) can be written

$$c_2 e^v + c_1^2 c_2 e^v e^{2u} + \left(2ac_1c_2^2 - 4ac_1c_2^2\right) e^{2v} e^u + \left(2c_2^3 - 4a^2c_2^3 - 4c_2^3\right) e^{3v} = 0$$

It is easy to see that the coefficients c_1 and c_2 have to be zero in order to satisfy the above equation, but this is not possible.

Then we have the following:

Corollary 3.1. *There does not exist an exponential affine translation surface with non-zero constant Gaussian curvature in* E^3 .

If we get the functions α and β as the exponential ones again, the equation (2.7) can be written

$$\begin{aligned}&\left[2c_1c_2^2e^ue^{2v}+c_2e^v\left(1+a^2+c_1^2e^{2u}\right)\right]\left[1+\left(c_1e^u+ac_2e^v\right)^2+c_2^2e^{2v}\right]\\&-3\left[\left(c_1e^u\left(1+c_2^2e^{2v}\right)+c_2e^v\left(1+a^2+c_1^2e^{2u}\right)\right)\right.\\&\quad\left.\left(\left(c_1e^u+ac_2e^v\right)ac_2e^v+c_2^2e^{2v}\right)\right]=0\end{aligned}$$

Considering the same technique mentioned before we obtain the following result:

Corollary 3.2. *There does not exist an exponential affine translation surface with non-zero constant mean curvature in* E^3 .

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