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Invariant subspaces of operators quasi-similar to L-weakly and M-weakly compact operators

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Abstract: Let T be an L-weakly compact operator defined on a Banach lattice E without order continuous norm. We prove that the bounded operator S defined on a Banach space X has a nontrivial closed invariant subspace if there exists an operator in the commutant of S that is quasi-similar to T. Additively, some similar and relevant results are extended to a larger classes of operators called super right-commutant. We also show that quasi-similarity need not preserve L-weakly or M-weakly compactness.

Key words: Invariant subspace, L-weakly compact operator, M-weakly compact operator, quasi-similarity

1. Introduction

The notion of quasi-similarity was first introduced by Sz.-Nagy and Foiaş in [8]. Following that, there has been considerable interest in quasi-similarity. If T is an operator that is quasi-similar to an operator with an invariant subspace, then it is not known if T needs to have an invariant subspace. However, the following theorem was proved in [5]:

If S and T are quasi-similar operators acting on the Hilbert spaces H and K respectively, and if S has a hyperinvariant subspace, then so does T. If, in addition, S is normal, then the lattice of hyperinvariant subspaces for T contains a sublattice that is lattice isomorphic to the lattice of spectral projections for S.

As is known, if E is a Banach lattice without order continuous norm and $E^a \neq \{0\}$, then L-weakly compact operators have a common nontrivial closed invariant subideal. Based on this, using the notion of quasi-similarity, we can consider the existence of nontrivial invariant subspaces for bounded operators on a Banach space X, which is different from E. For this reason, the purpose of this paper is to present invariant subspaces of bounded operators quasi-similar to some L-weakly or M-weakly compact operators defined on Banach lattices in terms without order continuous norm or dual norm.

In this paper, X and Y are infinite-dimensional Banach spaces while E and F denote infinite-dimensional Banach lattices. The positive cone of E will be denoted by E^+ and we will write $\mathcal{L}(X,Y)$, $\mathcal{W}_L(X,E)$, and $\mathcal{W}_M(X,E)$ for the bounded operators, L-weakly compact operators, and M-weakly compact operators respectively. We use the abbreviations $\mathcal{L}(X,X)=\mathcal{L}(X)$, $\mathcal{W}_L(E,E)=\mathcal{W}_L(E)$, and $\mathcal{W}_M(E,E)=\mathcal{W}_M(E)$. The commutant of an operator $S\in\mathcal{L}(X)$ is

$$\{S\}' = \{R\in\mathcal{L}(X) : SR=RS\}.$$

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The super right-commutant and super left-commutant of an operator $S \in \mathcal{L}^+(F)$ are

$$[S\rangle = \left\{B \in \mathcal{L}^+(F) : SB \leq BS\right\} \quad \text{and} \quad \langle S] = \left\{B \in \mathcal{L}^+(F) : SB \geq BS\right\},$$

respectively.

A closed subspace $U \subset X$ is a nontrivial invariant closed subspace under $T \in \mathcal{L}(X)$ (or nontrivial closed T-invariant) if $\{0\} \neq U \neq X$ and $T(U) \subseteq U$. Also, U is a T-hyperinvariant subspace if U is invariant under every operator that commutes with T. For $T \in \mathcal{L}(X)$ and for $0 \neq x \in X$, the linear span of $\left\{x, Tx, T^2x, T^3x, ...\right\}$ is denoted by $O_T(x)$ and is called the T-orbit space of x. If $\overline{O_T(x)} \neq X$ for some $0 \neq x \in X$ then $\overline{O_T(x)}$ is a nontrivial closed T-invariant subspace. Also, trivially, $\overline{RangeT}$ and $KerT$ are closed T-hyperinvariant subspaces. For the subspace U of a Banach lattice if $|x| \leq |y|$ and $y \in U$ imply $x \in U$ then U is called an ideal.

L-weakly and M-weakly compactnesses were introduced by Meyer-Nieberg in [6]. Recall that a nonempty bounded subset A of Banach lattice E is said to be L-weakly compact if $\|x_n\| \to 0$ as $n \to \infty$ for every disjoint sequence (x_n) in the solid hull of A. A bounded linear operator $T : X \to E$ is called L-weakly compact if $T(B_X)$ is L-weakly compact in E, where B_X denotes the closed unit ball of X. A bounded linear operator $T : E \to X$ is M-weakly compact if $\|Tx_n\| \to 0$ as $n \to \infty$ for every disjoint sequence (x_n) in B_E. In [6], it was shown that an operator defined between two Banach lattices is L-weakly (M-weakly) compact if and only if its adjoint operator is M-weakly (L-weakly) compact. Also, it is indicated that L-weakly compact and M-weakly compact operators are weakly compact operators. In general, L-weakly (or M-weakly) compact operators and compact operators are different classes.

An operator $P \in \mathcal{L}(X, Y)$ is a *quasi-affinity* if it is injective and has dense range. An operator $S \in \mathcal{L}(X)$ is said to be a *quasi-affine transform* of an operator $T \in \mathcal{L}(Y)$ if there exists a quasi-affinity $P \in \mathcal{L}(X, Y)$ such that $TP = PS$. The operators $S \in \mathcal{L}(X)$ and $T \in \mathcal{L}(Y)$ are *quasi-similar*, denoted by $S \overset{qs}{\backsim} T$, if there exist quasi-affinities $P \in \mathcal{L}(X, Y)$ and $Q \in \mathcal{L}(Y, X)$ such that $TP = PS$ and $QT = SQ$. If $T \in \mathcal{L}^+(E)$, $S \in \mathcal{L}^+(F)$, $P \in \mathcal{L}^+(F, E)$, $Q \in \mathcal{L}^+(E, F)$ then $S \in \mathcal{L}(F)$ and $T \in \mathcal{L}(E)$ are *positively quasi-similar*, denoted by $S \overset{pqs}{\backsim} T$ ([4, Definition 2.1]). Quasi-similarity is an equivalence relation on the class of all operators.

We refer to [2, 7] for notations and terminology concerning Banach lattices and operators on them and [1] for further details on the invariant subspace problem.

2. Auxiliary results

A Banach lattice E has an order continuous norm if $x_\alpha \downarrow 0$ in E implies $\|x_\alpha\| \downarrow 0$. All separable Dedekind complete Banach lattices have order continuous norm but ℓ_∞ and c (with the sup norm) are the best known examples of Banach lattices without order continuous norms. The order continuous part of a Banach lattice E is $E^a = \{x \in E : |x| \geqslant x_\alpha \downarrow 0 \Rightarrow \|x_\alpha\| \to 0\}$. For example, $(\ell^\infty)^a = c_0$ and $(L^\infty(\mu))^a = \{0\}$ where μ is a measure without atom. E^a is a closed order ideal and contains all L-weakly compact subsets of E ([7, Proposition 2.4.10, Proposition 3.6.2]).

Suppose that $E \neq E^a$ and $E^a \neq \{0\}$. The equality $E^a = \{0\}$ is equivalent to the fact that the zero operator is a unique E-valued L-weakly compact operator, and so considering such type of operators it is natural to assume $E^a \neq \{0\}$. Since L-weakly compact sets are contained in E^a then $RangeT \subset E^a$ for $0 \neq T \in \mathcal{W}_L(E)$ ([2, Theorem 5.66]). Therefore, $\overline{RangeT}$ is a nontrivial closed T-hyperinvariant subspace.

More generally, we can state that a bounded operator that commutes with some L-weakly compact operator defined on a Banach lattice without order continuous norm has a nontrivial closed invariant subspace. Can we extend this observation to a larger class of operators?

$J \subset \mathcal{L}(E)$ is called a two-sided ideal if $ST \in J$ and $TS \in J$ for $S,T \in J$. It is well known that $TS \in \mathcal{W}_L(E)$ always holds for $S \in \mathcal{L}(E)$ and for $T \in \mathcal{W}_L(E)$. However, $\mathcal{W}_L(E)$ and $\mathcal{W}_M(E)$ need not be two-sided ideals in $\mathcal{L}(E)$ (or in $\mathcal{L}^r(E)$) ([3, Example 1.2]). In [3], it was proved that $\mathcal{W}_L(E) \cap \mathcal{L}^r(E)$ is a two-sided ideal in $\mathcal{L}^r(E)$ if and ony if E has an order continuous norm. As a dual version, $\mathcal{W}_M(E) \cap \mathcal{L}^r(E)$ is a two-sided ideal in $\mathcal{L}^r(E)$ if and only if the dual E' has an order continuous norm.

Theorem 2.1 *Let E be a Banach lattice such that $E \neq E^a \neq \{0\}$. If $0 \neq T \in \mathcal{L}(E)$ and $0 \neq S \in \mathcal{L}(E)$ such that $S^k T \in \mathcal{W}_L(E)$ for $k = 1, 2, ...$ then S has a nontrivial closed invariant subspace.*

Proof Let us choose a nonzero element $x \in E$ such that $Tx \neq 0$. If $STx = 0$ then $KerS$ is a closed S-hyperinvariant subspace. Assume that $STx \neq 0$ and $S^kT \in \mathcal{W}_L(E)$ for $k = 1, 2, ...$. We have $S^kTx \in E^a$ for $k = 1, 2, ...$. Therefore, the closed subspace generated by the set $\{STx, S^2Tx, ..., S^kTx, ...\}$ is a nontrivial closed S-invariant subspace. □

Note that the class of operators S covered in the above theorem is larger than $\mathcal{W}_L(E)$, the commutant $\{T\}'$ for $T \in \mathcal{W}_L(E)$, and the algebra generated by $T \in \mathcal{W}_L(E)$.

On the other hand, it is natural to ask if quasi-similarity preserves L-weakly and M-weakly compactness. In order to answer that question we first will describe operators that are quasi-similar to a finite-rank operator. We write $f \otimes u$ for the rank one operator $x \to f(x)u$ if $f \in E^\sim$ and $u \in F$. Every operator $T : E \to F$ of the form $T = \sum_{i=1}^n f_i \otimes u_i$, where $f_i \in E^\sim$ and $u_i \in F$ $(i = 1, 2, ..., n)$, is called a finite rank operator and the collection of all finite rank operators from E to F will be denote by $E^\sim \otimes F$.

Proposition 2.2 *If $T \in F^\sim \otimes F$ and T is quasi-similar to $S \in \mathcal{L}(E)$ then $S \in E^\sim \otimes E$ and $rank(T) = rank(S)$.*

Proof Let $T = \sum_{i=1}^n f_i \otimes u_i$ for $\exists n \in \mathbb{N}$, $f_i \in F^\sim$ and linear independent elements $u_i \in F$ $(1 \leq i \leq n)$. If T is quasi-similar to S then there exist quasi-affinities $P \in \mathcal{L}(E, F)$ and $Q \in \mathcal{L}(F, E)$ such that $TP = PS$ and $QT = SQ$. For every $x \in E$,

$$QTx = SQx \Longrightarrow \sum_{i=1}^n f_i(x) Qu_i = SQx.$$

It follows that $RangeS = S\left(\overline{RangeQ}\right) \subseteq \overline{RangeSQ} \subseteq sp\{Qu_1, Qu_2, ..., Qu_n\}$. It means that S is a finite rank operator. Furthermore, $rank(S) \leq n = rank(T)$ holds, so by symmetry we have $rank(T) \leq rank(S)$, that is, $rank(T) = rank(S)$. □

Remark 2.3 *Suppose that $T = f \otimes u \in \mathcal{L}(E)$ for $f \in E'$, $u \in E$ and T is quasi-similar to $S \in \mathcal{L}(F)$. Then there exists a quasi-affinity $Q : E \to F$ such that $QT = SQ$, so $RangeS \subseteq sp\{Qu\}$ holds. Hence, there exists a representation of S such that $S = g \otimes Qu$ for $g \in F'$. In this case, we have $Q'g = f$ since the equality*

$$SQx = g(Qx)Qu = f(x)Qu = QTx$$

holds for $x \in E$.

Corollary 2.4 *Quasi-similarity need not preserve L-weakly compactness (hence M-weakly compactness).*

Proof For the Banach lattice E, we may find quasi-affinities $P : E \to E$ and $Q : E \to E$ such that $PQ = I_E$ and $Q'P' = I_{E'}$ where I is identity operator. Let us choose an element $u \in E^a$ such that $Qu \notin E^a$. If the operators $T, S \in \mathcal{L}(E)$ are defined by $T = f \otimes u$ and $S = P'f \otimes Qu$, respectively, then it is easy to see that T is quasi-similar to S. However, T is an L-weakly compact operator while S is not. □

3. Quasi-similarity to L-weakly compact operators

In this section, we investigate the applicability of Theorem 2.1 in the previous section for some classes of bounded operators on a Banach space by the help of quasi-affinities.

Theorem 3.1 *Let X be a Banach space, let E be a Banach lattice without order continuous norm such that $E^a \neq \{0\}$, and let $S \in \mathcal{L}(X)$. Suppose that there exists $T \in \mathcal{L}(E)$ such that:*

1. *There exists a polynomial p such that $0 \neq p(T) \in \mathcal{W}_L(E)$.*
2. *There exists $0 \neq R \in \{S\}'$, which is a quasi-affine transform of T.*

Then S has a nontrivial closed invariant subspace.

Proof Let us choose a nonzero operator $R \in \{S\}'$, which is a quasi-affine transform of T. Then there exists a quasi-affinity $P \in \mathcal{L}(X, E)$ such that $TP = PR$. Hence, $p(R)$ is a quasi-affine transform of $p(T)$ such that $p(T)P = Pp(R)$. Therefore, $Range(Pp(R)) = Range(p(T)P) \subseteq Range(p(T)) \subseteq E^a$. This yields that $Range(p(R))$ is not dense. If $Range(p(R)) = \{0\}$ then $p(T) = 0$ holds since P has dense range. This contradicts the assumption $p(T) \neq 0$. Then $\overline{Range(p(R))}$ is a nontrivial closed hyperinvariant subspace for $p(R)$, so S has a nontrivial closed invariant subspace since S also commutes with $p(R)$. □

Theorem 3.2 *Let X be a Banach space, let E be a Banach lattice such that $E \neq E^a \neq \{0\}$, and let $U \in \mathcal{L}(X)$. Suppose that there exists $0 \neq S \in \mathcal{L}(E)$ such that:*

1. *There exists $0 \neq T \in \mathcal{L}(E)$ such that $ST \in \mathcal{W}_L(E)$.*
2. *S' is injective.*
3. *There exists $0 \neq R \in \{U\}'$, which is a quasi-affine transform of T.*

Then $U \in \mathcal{L}(X)$ has a nontrivial closed invariant subspace.

Proof If $R \in \{U\}'$ is a quasi-affine transform of T then there exists a quasi-affinity $P \in \mathcal{L}(X, E)$ such that $TP = PR$ and for each $k = 1, 2, ...,$ $RU^k = U^k R$ holds. Let us choose a nonzero element $x \in E$ such that $Rx \neq 0$ since $R \neq 0$. If there exists a $k_0 \in \mathbb{N} - \{0\}$ such that $U^{k_0}Rx = 0$ then the closure of the subspace generated by the set $\{Rx, URx, U^2Rx, ..., U^{k_0-1}Rx\}$ is a nontrivial closed U-invariant subspace. Assume that $U^k Rx \neq 0$ for $k = 1, 2, ...$. If $ST \in \mathcal{W}_L(E)$ then for $k = 1, 2, ...$ we get $STPU^k \in \mathcal{W}_L(X, E)$, so $STPU^k x \in E^a$ since L-weakly compact subsets are contained in E^a. On the other hand, since E does not have order contiuous norm, according to seperating theorem, there exists $0 \neq f \in E'$ such that f is zero on E^a. Since P' and S' are injective $0 \neq P'S'f \in X'$ holds. It follows that for $k = 1, 2, ...$

$$\langle P'S'f, U^k Rx \rangle = \langle P'S'f, RU^k x \rangle = \langle f, SPRU^k x \rangle = \langle f, STPU^k x \rangle = 0$$

holds. This equality shows that the closed U-invariant subspace generated by the set $\{Rx, URx, U^2Rx, ..., U^kRx, ...\}$ is nontrivial. □

Theorem 3.3 *Let E and F be Banach lattices such that E has not order continuous norm and $E^a \neq \{0\}$. For $S \in \mathcal{L}^+(F)$ and $T \in \mathcal{W}_L^+(E)$, if there exists $B \in \mathcal{L}^+(F)$ such that:*

1. *$0 \neq B \in [S\rangle$,*
2. *There exists a positive quasi-affinity $P \in \mathcal{L}^+(F, E)$ such that $TP \geq PB$,*

then S has a nontrivial closed invariant ideal.

Proof We prove this using similar techniques to Theorem 10.24 in [1]. If $B \in [S\rangle$ then $SB \leq BS$, so $S^k B \leq BS^k$ holds for each $k \in \mathbb{N}$. Without loss of generality, we can assume that $\|S\| < 1$, which implies that the series $A = \sum_{n=0}^{\infty} S^n$ converges and defines a positive operator on F, which in turn implies $AB \leq BA$. Let choose $0 \neq x \in F$ such that $Bx \neq 0$. If $ABx = 0$ then the closure of the principal ideal generated by Bx is a nontrivial closed S-invariant ideal. Suppose that $ABx \neq 0$. If I is the principal ideal generated by ABx, i.e. $I = \{y \in F : \text{there exists } \lambda \geq 0 \text{ such that } |y| \leq \lambda ABx\}$ then $I \neq \{0\}$ and I is S-invariant since the inequalities $|Sy| \leq S|y| \leq S(\lambda ABx) = \lambda \sum_{n=1}^{\infty} S^n Bx \leq \lambda ABx$ hold for $y \in I$. As E does not have an order continuous norm, we have $0 \neq f \in (E')^+$ such that f is zero on E^a. Since P has dense range the adjoint operator P' is injective, so $P'f \neq 0$. Since $TPAx \in E^a$ for any $y \in I$

$$0 \leq |P'f(y)| \leq P'f|y| \leq P'f(\lambda ABx) = \lambda f(PBAx) \leq \lambda f(TPAx) = 0$$

holds. It follows that $\overline{I} \neq F$. Note that if $Ax = 0$ then the principal ideal generated by x is a nontrivial closed S-invariant ideal. □

4. Quasi-similarity to M-weakly compact operators

If A is a subset of Banach lattice E, then its polar A° is defined by $A^\circ = \{x' \in E' : |x'(x)| \leq 1 \text{ for every } x \in A\}$. A° is a convex, circled, and $\sigma(E', E)$-closed subset. If B is a subset of the dual space E' then

$$^\circ B = \{x \in E : |x'(x)| \leq 1 \text{ for every } x' \in B\}$$

is called the prepolar of B. If $B \subseteq E'$ is an ideal, then $^\circ B$ is an ideal, which is

$$^\circ B = \{x \in E : x'(x) = 0 \text{ for every } x' \in B\}.$$

According to definitions we have $A \subseteq {}^\circ(A^\circ)$ and $B \subseteq ({}^\circ B)^\circ$ ([2, Theorem 9.17]).

There are some situations where the prepolar ${}^{\circ}(E')^{a}$ is not equal to $\{0\}$ for the Banach lattice E. If the inclusion $(E')^{a} \subseteq E_{n}^{\sim}$ holds and $(E')^{a}$ is not order dense in $E_{n}^{\sim}$, then ${}^{\circ}(E')^{a} \neq \{0\}$ holds ([9, Corollary 105.12]). It is well known that if E has order continuous norm then $E' = E_{n}^{\sim}$ holds. For instance, the Banach lattice $E = L^{1}[0,1] \oplus c_{0}$ has order continuous norm. On the other hand, $E' = L^{\infty}[0,1] \oplus \ell_{1}$ does not have order continuous norm and $(E')^{a} = \ell_{1}$ is not order dense in E'. On the contrary, the ideal $(\ell_{1}')^{a} = c_{0}$ is order dense in $(\ell_{1})' = \ell_{\infty}$.

Theorem 4.1 *Let X be a Banach space and let E be a Banach lattice such that ${}^{\circ}(E')^{a} \neq \{0\}$. For $0 \neq S \in \mathcal{L}(X)$, if there exists $0 \neq T \in \mathcal{W}_{M}(E)$, which is a quasi-affine transform of S, then $R \in \{S\}'$ has a nontrivial closed invariant subspace.*

Proof Since T is a quasi-affine transform of S there exists a quasi-affinity $Q \in \mathcal{L}(E, X)$ such that $SQ = QT$. Since $QT \in \mathcal{W}_{M}(E, X)$ we have $T'Q' \in \mathcal{W}_{L}(X', E')$, so $T'Q'f \in (E')^{a}$ for any $f \in X'$. It follows that for $0 \neq x \in {}^{\circ}(E')^{a}$

$$\langle f, SQx\rangle = \langle f, QTx\rangle = \langle T'Q'f, x\rangle = 0.$$

Since $\langle X, X'\rangle$ is a dual pair we have $SQx = 0$ for $x \in {}^{\circ}(E')^{a}$. Since Q is injective $Qx \neq 0$, so since $S \neq 0$, $KerS$ is a nontrivial closed S-hyperinvariant subspace. Hence, $0 \neq R \in \{S\}'$ has a nontrivial closed invariant subspace. □

Corollary 4.2 *Let X be a Banach space and let E be a Banach lattice such that ${}^{\circ}(E')^{a} \neq \{0\}$. For $0 \neq S \in \mathcal{L}(X)$, if there exists $0 \neq T \in \mathcal{W}_{M}(E)$ which is a quasi-affine transform of some $0 \neq R \in \{S\}'$, then S has a nontrivial closed invariant subspace.*

Proof If $0 \neq R \in \{S\}'$ then $S \in \{R\}'$ holds. Thus, the corollary follows from previous theorem. □

Corollary 4.3 *Let X be a Banach space and let E be a Banach lattice such that ${}^{\circ}(E')^{a} \neq \{0\}$. If $S \in \mathcal{L}(X)$ and $T \in \mathcal{W}_{M}(E)$ such that T is a quasi-affine transform of $S - \lambda I$ for $0 \neq \lambda \in \mathbb{R}$ where I is identity operator on X, then S has a nonzero eigenvector or S is a scalar operator.*

Proof Under these assumptions, from the proof of Theorem 4.1 we see that there exist $0 \neq x \in {}^{\circ}(E')^{a}$ and a quasi-affinity $Q \in \mathcal{L}(E, X)$, which implies $Qx \neq 0$ such that $(S - \lambda I)Qx = 0$. Otherwise, if the subspace generated by the set $\{Qx : x \in {}^{\circ}(E')^{a}\}$ is dense in X then $S - \lambda I = 0$, so this means that S is a scalar operator. □

Corollary 4.4 *Let X be a Banach space and let E be a Banach lattice such that $E' \neq (E')^{a} \neq \{0\}$. Assume that $0 \neq S \in \mathcal{L}(X)$ and:*

1. *S is weakly compact and S'' is injective.*
2. *There exists $T \in \mathcal{W}_{M}(E)$ such that T' is a quasi-affine transform of S'.*

Then S has a nontrivial closed invariant subspace.

Proof If T' is a quasi-affine transform of S', then there exists a nontrivial closed S'-invariant subspace $V \subset X'$ by Theorem 3.1. Hence, there exist $0 \neq x'' \in X''$ such that $x'' = 0$ on V. Since S is a weakly compact operator, $S''(X'') \subseteq X$ holds by Gantmacher's theorem, so $x = S''(x'') \in X$. By the injectivity of S'' we get $W = \overline{sp\{S^k x : k \in \mathbb{N}\}} \neq \{0\}$ and clearly W is a closed S-invariant subspace. For $0 \neq g \in V$ and for $k \in \mathbb{N}$ the equivalent

$$\langle g, S^k x\rangle = \left\langle (S')^k g, x\right\rangle = \left\langle (S')^k g, S'' x''\right\rangle = \left\langle (S')^{k+1} g, x''\right\rangle = 0$$

shows that $W \neq X$. □

Theorem 4.5 *Let E and F be Banach lattices such that $^{\circ}(E')^a \neq \{0\}$. For $0 \neq S \in \mathcal{L}^+(F)$, $0 \neq T \in \mathcal{W}_M^+(E)$, and a positively quasi-affinity $Q \in \mathcal{L}^+(E,F)$, if $SQ \leq QT$ holds then every nonzero $R \in [S\rangle$ has a nontrivial closed invariant ideal.*

Proof For $0 \neq x \in {}^{\circ}(E')^a$ injectivity of Q implies $Qx \neq 0$. For $0 \neq f \in E'$, $T'Q'|f| \in (E')^a$ since $QT \in \mathcal{W}_M(E,F)$, so we obtain that

$$|\langle f, SQx\rangle| \leq \langle |f|, QT|x|\rangle = \langle T'Q'|f|, |x|\rangle = 0.$$

It follows that $SQx = 0$ for $x \in {}^{\circ}(E')^a$ since $\langle X, X'\rangle$ is a dual pair. For $0 \neq R \in [S\rangle$, let W be the closure of the ideal generated by the set $\{Qx, RQx, R^2Qx, ...\}$. Clearly, $W \neq \{0\}$ and clearly W is R-invariant. If $S \neq 0$ then $S' \neq 0$, so there exists $0 \neq f \in X'$ such that $S'f \neq 0$. Thus, since $SQ|x| = 0$, we get

$$\left|\langle S'f, R^kQx\rangle\right| \leq \langle |f|, SR^kQ|x|\rangle \leq \langle |f|, R^kSQ|x|\rangle = \langle |f|, R^k 0\rangle = \langle |f|, 0\rangle = 0$$

for $k \in \mathbb{N}$. This shows that $W \neq X$. □

Corollary 4.6 *Let E and F be Banach lattices such that $^{\circ}(E')^a \neq \{0\}$. For $0 \neq S \in \mathcal{L}^+(F)$, $0 \neq T \in \mathcal{W}_M^+(E)$, and a positively quasi-affinity $Q \in \mathcal{L}^+(E,F)$, if there exists $0 < R \in \langle S]$ such that $RQ \leq QT$, then S has a nontrivial closed invariant ideal.*

Proof If $0 \neq R \in \langle S]$ then $S \in [R\rangle$, so it follows from the previous theorem. □

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