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SPECTRAL PROBLEM FOR THE STURM–LIOUVILLE OPERATOR WITH RETARDED ARGUMENT CONTAINING A SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

E. Şen[1], **M. Acikgoz**[2], **and S. Araci**[3]

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We consider a discontinuous Sturm–Liouville problem with retarded argument containing a spectral parameter in the boundary condition. First, we investigate the simplicity of eigenvalues and then prove the existence theorem. As a result, we obtain the asymptotic formulas for eigenvalues and eigenfunctions.

1. Preliminaries

The boundary-value problems for differential equations of the second order with retarded argument were studied in [1–9]. Various physical applications of these problems can be found in [2]. The asymptotic formulas for the eigenvalues and eigenfunctions of boundary-value problems of the Sturm–Liouville type for the second-order differential equations with retarded argument were obtained in [1, 2, 5–9]. The asymptotic formulas for the eigenvalues and eigenfunctions of the classical Sturm–Liouville problem with spectral parameter in the boundary condition were obtained in [10–13].

In the present paper, we study the eigenvalues and eigenfunctions of a discontinuous boundary-value problem with retarded argument and spectral parameter in the boundary condition. This means that we consider a boundary-value problem for the differential equation

$$p(x)y''(x) + q(x)y(x - \Delta(x)) + \lambda y(x) = 0 \tag{1.1}$$

on $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$, with boundary conditions

$$y'(0) = 0, \tag{1.2}$$

$$y'(\pi) + \lambda y(\pi) = 0, \tag{1.3}$$

and jump conditions

$$\gamma_1 y(r_1 - 0) = \delta_1 y(r_1 + 0), \tag{1.4}$$

$$\gamma_2 y'(r_1 - 0) = \delta_2 y'(r_1 + 0), \tag{1.5}$$

$$\theta_1 y(r_2 - 0) = \eta_1 y(r_2 + 0), \tag{1.6}$$

[1] N. Kemal University, Tekirdağ, Turkey.
[2] Gaziantep University, Gaziantep, Turkey.
[3] H. Kalyoncu University, Gaziantep, Turkey.

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$$\theta_2 y'(r_2 - 0) = \eta_2 y'(r_2 + 0), \tag{1.7}$$

where $p(x) = p_1^2$ for $x \in [0, r_1)$, $p(x) = p_2^2$ for $x \in (r_1, r_2)$, and $p(x) = p_3^2$ for $x \in (r_2, \pi]$; the real-valued function $q(x)$ is continuous in $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$ and has the finite limits

$$q(r_1 \pm 0) = \lim_{x \to r_1 \pm 0} q(x) \quad \text{and} \quad q(r_2 \pm 0) = \lim_{x \to r_2 \pm 0} q(x);$$

the real valued function $\Delta(x) \geq 0$ is continuous in $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$ and has the finite limits

$$\Delta(r_1 \pm 0) = \lim_{x \to r_1 \pm 0} \Delta(x) \quad \text{and} \quad \Delta(r_2 \pm 0) = \lim_{x \to r_2 \pm 0} \Delta(x);$$

$x - \Delta(x) \geq 0$ for $x \in \left[0, \frac{\pi}{2}\right)$; $x - \Delta(x) \geq \frac{\pi}{2}$ for $x \in \left(\frac{\pi}{2}, \pi\right]$; λ is a real spectral parameter; $p_1,\ p_2,\ p_3,$ $\gamma_1,\ \gamma_2,\ \delta_1,\ \delta_2,\ \theta_1,\ \theta_2,\ \eta_1,$ and η_2 are arbitrary real numbers; $|\gamma_i| + |\delta_i| \neq 0$, and $|\theta_i| + |\eta_i| \neq 0$ for $i = 1, 2$. Moreover, the equalities $\gamma_1 \delta_2 p_1 = \gamma_2 \delta_1 p_2$ and $\theta_1 \eta_2 p_2 = \theta_2 \eta_1 p_3$ are true.

It is worth noting that some problems with jump conditions encountered in mechanics (a problem with thermal condition for a thin laminated plate) were studied in [14].

Let $w_1(x, \lambda)$ be a solution of Eq. (1.1) on $[0, r_1]$ satisfying the initial conditions

$$w_1(0, \lambda) = 1, \qquad w_1'(0, \lambda) = 0. \tag{1.8}$$

Conditions (1.8) define a unique solution of Eq. (1.1) on $[0, r_1]$ [2, p. 12].

After defining the indicated solution we define the solution $w_2(x, \lambda)$ of Eq. (1.1) on $[r_1, r_2]$ by using the solution $w_1(x, \lambda)$ with the initial conditions

$$w_2(r_1, \lambda) = \gamma_1 \delta_1^{-1} w_1(r_1, \lambda), \qquad w_2'(r_1, \lambda) = \gamma_2 \delta_2^{-1} \omega_1'(r_1, \lambda). \tag{1.9}$$

Conditions (1.9) are determined as a unique solution of Eq. (1.1) on $[r_1, r_2]$.

After defining the indicated solution, we determine the solution $w_3(x, \lambda)$ of Eq. (1.1) on $[r_2, \pi]$ by using the solution $w_2(x, \lambda)$ and the initial conditions

$$w_3(r_2, \lambda) = \theta_1 \eta_1^{-1} w_2(r_2, \lambda), \qquad w_3'(r_2, \lambda) = \theta_2 \eta_2^{-1} \omega_2'(r_2, \lambda). \tag{1.10}$$

Conditions (1.10) are defined as the unique solution of Eq. (1.1) on $[r_2, \pi]$.

Consequently, the function $w(x, \lambda)$ defined on $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$ by the equality

$$w(x, \lambda) = \begin{cases} w_1(x, \lambda), & x \in [0, r_1), \\ w_2(x, \lambda), & x \in (r_1, r_2), \\ w_3(x, \lambda), & x \in (r_2, \pi], \end{cases}$$

is a solution of Eq. (1.1) on $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$ satisfying one of the boundary conditions and both transmission conditions.

Lemma 1.1. *Let $w(x, \lambda)$ be a solution of Eq. (1.1) and let $\lambda > 0$. Then the following integral equations are true:*

$$w_1(x,\lambda) = \cos\frac{s}{p_1}x - \frac{1}{s}\int\limits_0^x \frac{q(\tau)}{p_1}\sin\frac{s}{p_1}(x-\tau)w_1\left(\tau-\Delta(\tau),\lambda\right)d\tau, \qquad s=\sqrt{\lambda}, \quad \lambda>0, \tag{1.11}$$

$$w_2(x,\lambda) = \frac{\gamma_1}{\delta_1}w_1\left(r_1,\lambda\right)\cos\frac{s}{p_2}\left(x-r_1\right) + \frac{\gamma_2 p_2 w_1'\left(r_1,\lambda\right)}{s\delta_2}\sin\frac{s}{p_2}\left(x-r_1\right)$$

$$-\frac{1}{s}\int\limits_{r_1}^x \frac{q(\tau)}{p_2}\sin\frac{s}{p_2}(x-\tau)w_2\left(\tau-\Delta\left(\tau\right),\lambda\right)d\tau, \qquad s=\sqrt{\lambda}, \quad \lambda>0, \tag{1.12}$$

$$w_3(x,\lambda) = \frac{\theta_1}{\eta_1}w_2\left(r_2,\lambda\right)\cos\frac{s}{p_3}\left(x-r_2\right) + \frac{\theta_2 p_3 w_2'\left(r_2,\lambda\right)}{s\eta_2}\sin\frac{s}{p_3}\left(x-r_2\right)$$

$$-\frac{1}{s}\int\limits_{r_2}^x \frac{q(\tau)}{p_3}\sin\frac{s}{p_3}(x-\tau)w_3\left(\tau-\Delta\left(\tau\right),\lambda\right)d\tau, \qquad s=\sqrt{\lambda}, \quad \lambda>0. \tag{1.13}$$

Proof. To prove this, it is sufficient to substitute

$$-\frac{s^2}{p_1^2}w_1(\tau,\lambda) - w_1''(\tau,\lambda), \qquad -\frac{s^2}{p_2^2}w_2(\tau,\lambda) - w_2''(\tau,\lambda),$$

and

$$-\frac{s^2}{p_3^2}w_3(\tau,\lambda) - w_3''(\tau,\lambda)$$

for

$$-\frac{q(\tau)}{p_1^2}w_1(\tau-\Delta(\tau),\lambda), \qquad -\frac{q(\tau)}{p_2^2}w_2(\tau-\Delta(\tau),\lambda), \qquad \text{and} \qquad -\frac{q(\tau)}{p_3^2}w_3(\tau-\Delta(\tau),\lambda)$$

in the integrals in (1.11), (1.12), and (1.13), respectively, and integrate these equations by parts twice.

Theorem 1.1. *Problem (1.1)–(1.7) may have only simple eigenvalues.*

Proof. Let $\widetilde{\lambda}$ be an eigenvalue of problem (1.1)–(1.7) and let

$$\widetilde{u}(x,\widetilde{\lambda}) = \begin{cases} \widetilde{u_1}(x,\widetilde{\lambda}), & x\in[0,r_1), \\ \widetilde{u_2}(x,\widetilde{\lambda}), & x\in(r_1,r_2), \\ \widetilde{u_3}(x,\widetilde{\lambda}), & x\in(r_2,\pi], \end{cases}$$

be the corresponding eigenfunction. Then it follows from (1.2) and (1.8) that the determinant

$$W\left[\widetilde{u}_1(0,\widetilde{\lambda}), w_1(0,\widetilde{\lambda})\right] = \begin{vmatrix} \widetilde{u}_1(0,\widetilde{\lambda}) & 1 \\ \widetilde{u}_1'(0,\widetilde{\lambda}) & 0 \end{vmatrix} = 0.$$

Moreover, by Theorem 2.2.2 in [2] the functions $\widetilde{u}_1(x,\widetilde{\lambda})$ and $w_1(x,\widetilde{\lambda})$ are linearly dependent on $[0,r_1]$. We can also prove that the functions $\widetilde{u}_2(x,\widetilde{\lambda})$ and $w_2(x,\widetilde{\lambda})$ are linearly dependent on $[r_1,r_2]$ and the functions $\widetilde{u}_3(x,\widetilde{\lambda})$ and $w_3(x,\widetilde{\lambda})$ are linearly dependent on $[r_2,\pi]$. Hence,

$$\widetilde{u}_i(x,\widetilde{\lambda}) = K_i w_i(x,\widetilde{\lambda}), \quad i = 1,2,3, \tag{1.14}$$

for some $K_1 \neq 0,\ K_2 \neq 0,$ and $K_3 \neq 0$. We first show that $K_2 = K_3$. Suppose that $K_2 \neq K_3$. It follows from equalities (1.6) and (1.14) that

$$\begin{aligned}\theta_1\widetilde{u}(r_2-0,\widetilde{\lambda}) - \eta_1\widetilde{u}(r_2+0,\widetilde{\lambda}) &= \theta_1\widetilde{u_2}(r_2,\widetilde{\lambda}) - \eta_1\widetilde{u_3}(r_2,\widetilde{\lambda})\\ &= \theta_1K_2w_2(r_2,\widetilde{\lambda}) - \eta_1K_3w_3(r_2,\widetilde{\lambda})\\ &= \theta_1K_2\eta_1\theta_1^{-1}w_3(r_2,\widetilde{\lambda}) - \eta_1K_3w_3(r_2,\widetilde{\lambda})\\ &= \eta_1\left(K_2-K_3\right)w_3(r_2,\widetilde{\lambda}) = 0.\end{aligned}$$

Since $\eta_1\left(K_2-K_3\right) \neq 0,$ we obtain

$$w_3\left(r_2,\widetilde{\lambda}\right) = 0. \tag{1.15}$$

By using the same procedure arising from (1.7), we conclude that

$$w_3'\left(r_2,\widetilde{\lambda}\right) = 0. \tag{1.16}$$

It follows from the fact that $w_3(x,\widetilde{\lambda})$ is a solution of the differential equation (1.1) on $[r_2,\pi]$ and satisfies the initial conditions (1.15) and (1.16) that $w_3(x,\widetilde{\lambda}) = 0$ identically on $[r_2,\pi]$ (cf. [2, p. 12], Theorem 1.2.1).

By using the same procedure, we can also find

$$w_1\left(r_1,\widetilde{\lambda}\right) = w_1'\left(r_1,\widetilde{\lambda}\right) = w_2\left(r_2,\widetilde{\lambda}\right) = w_2'\left(r_2,\widetilde{\lambda}\right) = 0.$$

Thus, we get

$$w_2(x,\widetilde{\lambda}) = 0 \quad \text{and} \quad w_1(x,\widetilde{\lambda}) = 0$$

identically on $[0,r_1)\cup(r_1,r_2)\cup(r_2,\pi]$. However, this contradicts (1.8), thus completing the proof.

2. Existence Theorem

The function $w(x,\lambda)$ defined in Sec. 1 is a nontrivial solution of Eq. (1.1) satisfying conditions (1.2), (1.4), (1.5), and (1.6). Substituing $w(x,\lambda)$ in (1.3), we arrive at the characteristic equation

$$F(\lambda) \equiv w'(\pi,\lambda) + \lambda w(\pi,\lambda) = 0. \tag{2.1}$$

By Theorem 1.1, the set of eigenvalues of the boundary-value problem (1.1)–(1.7) coincides with the set of real roots of Eq. (2.1). Let

$$q_1 = \frac{1}{p_1}\int\limits_0^{r_1} |q(\tau)|\, d\tau, \qquad q_2 = \frac{1}{p_2}\int\limits_{r_1}^{r_2} |q(\tau)|\, d\tau, \qquad \text{and} \qquad q_3 = \frac{1}{p_3}\int\limits_{r_2}^{\pi} |q(\tau)|\, d\tau.$$

Lemma 2.1.

(1) Let $\lambda \geq 4q_1^2$*. Then, for the solution* $w_1(x,\lambda)$ *of Eq. (1.11), the following inequality holds:*

$$|w_1(x,\lambda)| \leq 2, \quad x \in [0, r_1]. \tag{2.2}$$

(2) Let $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$*. Then, for the solution* $w_2(x,\lambda)$ *of Eq. (1.12), the following inequality holds:*

$$|w_2(x,\lambda)| \leq 4\left(\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_2}\right|\right), \quad x \in [r_1, r_2]. \tag{2.3}$$

(3) Let $\lambda \geq \max\left\{4q_1^2, 4q_2^2, 4q_3^2\right\}$*. Then, for the solution* $w_2(x,\lambda)$ *of Eq. (1.13), the following inequality holds:*

$$|w_3(x,\lambda)| \leq \frac{8\theta_1 p_2 + 4\theta_2 p_3 \eta_1}{\eta_1 p_2 \eta_2}\left(\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_2}\right|\right) + \frac{\theta_2 p_3}{\eta_2}\left|\frac{4\gamma_1\delta_2 q_1 + \gamma_2 p_2 \delta_1}{2p_2\delta_1\delta_2 q_1}\right|, \quad x \in [r_2, \pi]. \tag{2.4}$$

Proof. Assume that

$$B_{1\lambda} = \max_{[0,r_1]} |w_1(x,\lambda)|.$$

Thus, it follows from (1.11) that the following inequality holds for any $\lambda > 0$:

$$B_{1\lambda} \leq 1 + \frac{1}{s} B_{1\lambda} q_1.$$

If $s \geq 2q_1,$ then we get (2.2). Differentiating (1.11) with respect to $x,$ we find

$$w_1'(x,\lambda) = -\frac{s}{p_1}\sin\frac{s}{p_1}x - \frac{1}{p_1^2}\int\limits_0^x q(\tau)\cos\frac{s}{p_1}(x-\tau)\, w_1(\tau - \Delta(\tau),\lambda)\, d\tau. \tag{2.5}$$

Taking into account (2.5) and (2.2), for $s \geq 2q_1,$ we arrive at the following inequality:

$$\frac{|w_1'(x,\lambda)|}{s} \leq \frac{2}{p_1}. \tag{2.6}$$

Let

$$B_{2\lambda} = \max_{[r_1,r_2]} |w_2(x,\lambda)|.$$

Then it follows from (1.12), (2.2), and (2.6) that the following inequality holds for $s \geq 2q_1$:

$$B_{2\lambda} \leq 4\left\{\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_2}\right|\right\}.$$

Hence, if

$$\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\},$$

then we get (2.3).

Differentiating (1.12) with respect to $x,$ we obtain

$$w_2'(x,\lambda) = -\frac{s\gamma_1}{p_2\delta_1} w_1\left(r_1,\lambda\right)\sin\frac{s}{p_2}\left(x-r_1\right) + \frac{\gamma_2 w_1'\left(r_1,\lambda\right)}{\delta_2}\cos\frac{s}{p_2}\left(x-r_1\right)$$

$$-\frac{1}{p_2^2}\int\limits_{r_1}^{x} q(\tau)\cos\frac{s}{p_2}(x-\tau)w_2(\tau-\Delta(\tau),\lambda)\,d\tau. \tag{2.7}$$

By virtue of (2.7) and (2.3), for $s \geq 2q_2,$ the following inequality is true:

$$\frac{\left|w_2'(x,\lambda)\right|}{s} \leq \frac{2\gamma_1}{p_2\delta_1} + \frac{\gamma_2}{2\delta_2 q_1} + \frac{2}{p_2}\left\{\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_2}\right|\right\}. \tag{2.8}$$

Let

$$B_{3\lambda} = \max_{[r_2,\pi]}\left|w_3\left(x,\lambda\right)\right|,$$

Thus, it follows from (1.13), (2.2), (2.3), and (2.8) that the following inequality holds for $s \geq 2q_3$:

$$B_{3\lambda} \leq \frac{8\theta_1 p_2 + 4\theta_2 p_3\eta_1}{\eta_1 p_2\eta_2}\left(\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_2}\right|\right) + \frac{\theta_2 p_3}{\eta_2}\left|\frac{4\gamma_1\delta_2 q_1 + \gamma_2 p_2\delta_1}{2p_2\delta_1\delta_2 q_1}\right|.$$

Hence, if

$$\lambda \geq \max\left\{4q_1^2, 4q_2^2, 4q_3^2\right\},$$

then we arrive at Eq. (2.4).

Theorem 2.1. *Problem (1.1)–(1.7) has an infinite set of positive eigenvalues.*

Proof. Differentiating (1.13) with respect to $x,$ we obtain

$$w_3'(x,\lambda) = -\frac{s\theta_1}{p_3\eta_1} w_2\left(r_2,\lambda\right)\sin\frac{s}{p_3}\left(x-r_2\right) + \frac{\theta_2 w_2'\left(r_2,\lambda\right)}{\eta_2}\cos\frac{s}{p_3}\left(x-r_2\right)$$

$$-\frac{1}{p_3^2}\int\limits_{r_2}^{x} q(\tau)\cos\frac{s}{p_3}(x-\tau)w_3(\tau-\Delta(\tau),\lambda)d\tau. \tag{2.9}$$

From (1.11)–(1.13), (2.1), (2.5), (2.7), and (2.9), we get

$$-\frac{s\theta_1}{p_3\eta_1}\left[\frac{\gamma_1}{\delta_1}\left(\cos\frac{sr_1}{p_1} - \frac{1}{sp_1}\int\limits_{0}^{r_1} q(\tau)\sin\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\cos\frac{s}{p_2}\left(r_2-r_1\right)\right.$$

$$+\frac{\gamma_2 p_2}{s\delta_2}\left(-\frac{s}{p_1}\sin\frac{sr_1}{p_1}-\frac{1}{p_1^2}\int\limits_0^{r_1} q(\tau)\cos\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\sin\frac{s}{p_2}(r_2-r_1)$$

$$\left.-\frac{1}{sp_2}\int\limits_{r_1}^{r_2} q(\tau)\sin\frac{s}{p_2}(r_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right]\sin\frac{s}{p_3}(\pi-r_2)$$

$$+\frac{\theta_2}{\eta_2}\left[-\frac{s\gamma_1}{p_2\delta_1}\left(\cos\frac{sr_1}{p_1}-\frac{1}{sp_1}\int\limits_0^{r_1} q(\tau)\sin\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\sin\frac{s}{p_2}(r_2-r_1)\right.$$

$$+\frac{\gamma_2}{\delta_2}\left(-\frac{s}{p_1}\sin\frac{sr_1}{p_1}-\frac{1}{p_1^2}\int\limits_0^{r_1} q(\tau)\cos\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\cos\frac{s}{p_2}(r_2-r_1)$$

$$\left.-\frac{1}{p_2^2}\int\limits_{r_1}^{r_2} q(\tau)\cos\frac{s}{p_2}(r_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right]\cos\frac{s}{p_3}(\pi-r_2)$$

$$-\frac{1}{p_3^2}\int\limits_{r_2}^{\pi} q(\tau)\cos\frac{s}{p_3}(\pi-\tau)\,w_3\left(\tau-\Delta\left(\tau\right),\lambda\right)d\tau$$

$$+\lambda\left\{\frac{\theta_1}{\eta_1}\left[\frac{\gamma_1}{\delta_1}\left(\cos\frac{sr_1}{p_1}-\frac{1}{sp_1}\int\limits_0^{r_1} q(\tau)\sin\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\cos\frac{s}{p_2}(r_2-r_1)\right.\right.$$

$$+\frac{\gamma_2 p_2}{s\delta_2}\left(-\frac{s}{p_1}\sin\frac{sr_1}{p_1}-\frac{1}{p_1^2}\int\limits_0^{r_1} q(\tau)\cos\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\sin\frac{s}{p_2}(r_2-r_1)$$

$$\left.-\frac{1}{sp_2}\int\limits_{r_1}^{r_2} q(\tau)\sin\frac{s}{p_2}(r_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right]\cos\frac{s}{p_3}(\pi-r_2)$$

$$+\frac{\theta_2 p_3}{s\eta_2}\left[-\frac{s\gamma_1}{p_2\delta_1}\left(\cos\frac{sr_1}{p_1}-\frac{1}{sp_1}\int\limits_0^{r_1} q(\tau)\sin\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\sin\frac{s}{p_2}(r_2-r_1)\right.$$

$$+\frac{\gamma_2}{\delta_2}\left(-\frac{s}{p_1}\sin\frac{sr_1}{p_1}-\frac{1}{p_1^2}\int\limits_0^{r_1} q(\tau)\cos\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\cos\frac{s}{p_2}(r_2-r_1)$$

$$\left.-\frac{1}{p_2^2}\int\limits_{r_1}^{r_2} q(\tau)\cos\frac{s}{p_2}(r_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right]\sin\frac{s}{p_3}(\pi-r_2)$$

$$-\frac{1}{sp_3}\int_{r_2}^{\pi} q(\tau)\sin\frac{s}{p_3}(\pi-\tau)\,w_3(\tau-\Delta(\tau),\lambda)\,d\tau\Bigg\}=0. \tag{2.10}$$

Let λ be sufficiently large. Then, by (2.2)–(2.4), Eq. (2.10) can be rewritten in the form

$$s\cos s\left(\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}\right)+O(1)=0. \tag{2.11}$$

Obviously, for large $s,$ Eq. (2.11) has an infinite set of roots. Thus, we arrive at the required result.

3. Asymptotic Formulas for Eigenvalues and Eigenfunctions

We now begin to study the asymptotic properties of eigenvalues and eigenfunctions. In what follows, we assume that s is sufficiently large. From (1.11) and (2.2), we get

$$w_1(x,\lambda)=O(1). \tag{3.1}$$

It follows from expressions (1.12) and (2.3) that

$$w_2(x,\lambda)=O(1). \tag{3.2}$$

By virtue of (1.13) and (2.4), we arrive at the following equation:

$$w_3(x,\lambda)=O(1). \tag{3.3}$$

The existence and continuity of the derivatives $w'_{1s}(x,\lambda)$ for $0\le x\le r_1,$ $|\lambda|<\infty,$ $w'_{2s}(x,\lambda)$ for $r_1\le x\le r_2,$ $|\lambda|<\infty,$ and $w'_{3s}(x,\lambda)$ for $r_2\le x\le\pi, |\lambda|<\infty$ follows from Theorem 1.4.1 in [2]:

$$\begin{aligned} &w'_{1s}(x,\lambda)=O(1), \quad x\in[0,r_1],\\ &w'_{2s}(x,\lambda)=O(1), \quad x\in[r_1,r_2],\\ &w'_{3s}(x,\lambda)=O(1), \quad x\in[r_2,\pi]. \end{aligned} \tag{3.4}$$

Theorem 3.1. *Let n be a natural number. For any sufficiently large $n,$ there is exactly one eigenvalue of problem (1.1)–(1.7) near*

$$\frac{(n+1/2)^2\,\pi^2}{\left(r_1/p_1+(r_2-r_1)/p_2+(\pi-r_2)/p_3\right)^2}.$$

Proof. We now consider the expression denoted by $O(1)$ in Eq. (2.11). If relations (3.1)–(3.4) are taken into account, then it can be shown by differentiation with respect to s that, for large $s,$ the derivative of this expression is bounded. We now show that, for large $n,$ only one root of (2.11) lies near each

$$\frac{(n+1/2)^2\,\pi^2}{\left(r_1/p_1+(r_2-r_1)/p_2+(\pi-r_2)/p_3\right)^2}.$$

Consider a function

$$\phi(s) = s\cos s\left(\frac{r_1}{p_1} + \frac{r_2 - r_1}{p_2} + \frac{\pi - r_2}{p_3}\right) + O(1).$$

Its derivative has the form

$$\phi'(s) = \cos s\left(\frac{r_1}{p_1} + \frac{r_2 - r_1}{p_2} + \frac{\pi - r_2}{p_3}\right)$$

$$- s\left(\frac{r_1}{p_1} + \frac{r_2 - r_1}{p_2} + \frac{\pi - r_2}{p_3}\right)\sin s\left(\frac{r_1}{p_1} + \frac{r_2 - r_1}{p_2} + \frac{\pi - r_2}{p_3}\right) + O(1)$$

and does not vanish for s close to sufficiently large n. Thus, our assertion follows from the Rolle theorem.

Let n be sufficiently large. In what follows, we denote by $\lambda_n = s_n^2$ the eigenvalue of problem (1.1)–(1.7) located near

$$\frac{(n + 1/2)^2\,\pi^2}{\left(r_1/p_1 + (r_2 - r_1)/p_2 + (\pi - r_2)/p_3\right)^2}.$$

We set

$$s_n = \frac{\left(n + \frac{1}{2}\right)\pi}{\left(\frac{r_1}{p_1} + \frac{r_2 - r_1}{p_2} + \frac{\pi - r_2}{p_3}\right)} + \delta_n.$$

It follows from (2.11) that

$$\delta_n = O\left(\frac{1}{n}\right).$$

Therefore, we obtain

$$s_n = \frac{\left(n + \frac{1}{2}\right)\pi}{\left(\frac{r_1}{p_1} + \frac{r_2 - r_1}{p_2} + \frac{\pi - r_2}{p_3}\right)} + O\left(\frac{1}{n}\right). \tag{3.5}$$

Relation (3.5) makes it possible to obtain asymptotic expressions for the eigenfunctions of problem (1.1)–(1.7). By (1.11), (2.5), and (3.1), we get

$$w_1(x,\lambda) = \cos\frac{sx}{p_1} + O\left(\frac{1}{s}\right), \tag{3.6}$$

$$w_1'(x,\lambda) = -\frac{s}{p_1}\sin\frac{sx}{p_1} + O(1). \tag{3.7}$$

By virtue of (1.12), (3.2), (3.6), and (3.7), we find

$$w_2(x,\lambda) = \frac{\gamma_1}{\delta_1}\cos\frac{s}{p_2}\left(\frac{r_1\left(p_2-p_1\right)}{p_1}+x\right)+O\left(\frac{1}{s}\right), \tag{3.8}$$

$$w_2'(x,\lambda) = -\frac{s\gamma_1}{\delta_1 p_2}\sin\frac{s}{p_2}\left(\frac{r_1\left(p_2-p_1\right)}{p_1}+x\right)+O(1). \tag{3.9}$$

In view of (1.13), (3.3), (3.8), and (3.9), we conclude that

$$w_3(x,\lambda) = \frac{\theta_1\gamma_1}{\eta_1\delta_1}\cos\frac{s}{p_3}\left(\frac{p_3\left(r_1\left(p_2-p_1\right)+p_1 r_2\right)-r_2 p_1 p_2}{p_1 p_2}+x\right)+O\left(\frac{1}{s}\right). \tag{3.10}$$

Substituting (3.5) in (3.6), (3.8), and (3.10), we immediately obtain

$$u_{1n}(x) = \cos\left(\frac{\left(n+\frac{1}{2}\right)\pi x}{p_1\left(\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}\right)}\right)+O\left(\frac{1}{n}\right),$$

$$u_{2n}(x) = \frac{\gamma_1}{\delta_1}\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi}{p_2\left(\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}\right)}\left(\frac{r_1\left(p_2-p_1\right)}{p_1}+x\right)\right)+O\left(\frac{1}{n}\right),$$

$$u_{3n}(x) = \frac{\theta_1\gamma_1}{\eta_1\delta_1}$$

$$\times\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi}{p_3\left(\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}\right)}\left(\frac{p_3(r_1\left(p_2-p_1\right)+p_1 r_2)-r_2 p_1 p_2}{p_1 p_2}+x\right)\right)+O\left(\frac{1}{n}\right).$$

Hence, the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x) = \begin{cases} u_{1n}(x) = w_1\left(x,\lambda_n\right), & x\in[0,r_1), \\ u_{2n}(x) = w_2\left(x,\lambda_n\right), & x\in\left(r_1,r_2\right), \\ u_{3n}(x) = w_3\left(x,\lambda_n\right), & x\in\left(r_2,\pi\right]. \end{cases}$$

Under certain additional conditions, we can obtain more exact asymptotic formulas depending on the delay. Assume that the following conditions are satisfied:

(a) the derivatives $q'(x)$ and $\Delta''(x)$ exist, are bounded in $[0,r_1)\cup(r_1,r_2)\cup(r_2,\pi]$, and have the following finite limits:

$$q'(r_1\pm 0) = \lim_{x\to r_1\pm 0} q'(x), \qquad q'(r_2\pm 0) = \lim_{x\to r_2\pm 0} q'(x),$$

$$\Delta''(r_1 \pm 0) = \lim_{x\to r_1\pm 0} \Delta''(x), \qquad \text{and} \qquad \Delta''(r_2 \pm 0) = \lim_{x\to r_2\pm 0} \Delta''(x),$$

respectively;

(b) $\Delta'(x) \le 1$ in $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$, $\Delta(0) = 0$, $\lim_{x\to r_1+0} \Delta(x) = 0$, and $\lim_{x\to r_2+0} \Delta(x) = 0$.

By using (b), we find

$$\begin{aligned} &x - \Delta(x) \ge 0 \quad \text{for} \quad x \in [0, r_1),\\ &x - \Delta(x) \ge r_1 \quad \text{for} \quad x \in (r_1, r_2),\\ &x - \Delta(x) \ge r_2 \quad \text{for} \quad x \in (r_2, \pi]. \end{aligned} \tag{3.11}$$

It follows from (3.6), (3.8), (3.10), and (3.11) that

$$\begin{aligned} &w_1(\tau - \Delta(\tau), \lambda) = \cos \frac{s\,(\tau - \Delta(\tau))}{p_1} + O\left(\frac{1}{s}\right),\\ &w_2(\tau - \Delta(\tau), \lambda) = \frac{\gamma_1}{\delta_1}\frac{\gamma_1}{\delta_1} \cos \frac{s}{p_2}\left(\frac{r_1\,(p_2 - p_1)}{p_1} + \tau - \Delta\,(\tau)\right) + O\left(\frac{1}{s}\right),\\ &w_3(\tau - \Delta(\tau), \lambda) = \frac{\theta_1\gamma_1}{\eta_1\delta_1} \cos \frac{s}{p_3}\left(\frac{p_3\,(r_1\,(p_2 - p_1) + p_1 r_2) - r_2 p_1 p_2}{p_1 p_2} + \tau - \Delta(\tau)\right) + O\left(\frac{1}{s}\right). \end{aligned} \tag{3.12}$$

Under the conditions (a) and (b), the formulas

$$O\left(\frac{1}{s}\right) = \begin{cases} \displaystyle\int_0^{r_1} \frac{q(\tau)}{2} \sin \frac{s}{p_1}\,(2\tau - \Delta(\tau))\,d\tau,\\ \displaystyle\int_0^{r_1} \frac{q(\tau)}{2} \cos \frac{s}{p_1}\,(2\tau - \Delta(\tau))\,d\tau,\\ \displaystyle\int_{r_1}^{r_2} \frac{q(\tau)}{2} \sin \frac{s}{p_2}\,(2\tau - \Delta(\tau))\,d\tau,\\ \displaystyle\int_{r_1}^{r_2} \frac{q(\tau)}{2} \cos \frac{s}{p_2}\,(2\tau - \Delta(\tau))\,d\tau,\\ \displaystyle\int_{r_2}^{\pi} \frac{q(\tau)}{2} \sin \frac{s}{p_3}\,(2\tau - \Delta(\tau))\,d\tau,\\ \displaystyle\int_{r_2}^{\pi} \frac{q(\tau)}{2} \cos \frac{s}{p_3}\,(2\tau - \Delta(\tau))\,d\tau \end{cases} \tag{3.13}$$

can be proved by using the same technique as in Lemma 3.3.3 from [2]. By using the notation

$$A(x)=\int\limits_0^x \frac{q(\tau)}{2}\sin\frac{s\Delta(\tau)}{p_1}d\tau, \qquad B(x)=\int\limits_0^x \frac{q(\tau)}{2}\cos\frac{s\Delta(\tau)}{p_1}d\tau,$$

$$C(x)=\int\limits_{r_1}^x \frac{q(\tau)}{2}\sin\frac{s\Delta(\tau)}{p_2}d\tau, \qquad D(x)=\int\limits_{r_1}^x \frac{q(\tau)}{2}\cos\frac{s\Delta(\tau)}{p_2}d\tau,$$

$$E(x)=\int\limits_{r_2}^x \frac{q(\tau)}{2}\sin\frac{s\Delta(\tau)}{p_3}d\tau, \qquad F(x)=\int\limits_{r_2}^x \frac{q(\tau)}{2}\cos\frac{s\Delta(\tau)}{p_3}d\tau,$$

$$Z_p^r=\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}, \qquad \Delta_p^r=\frac{1}{p_3}+\frac{B(r_1)}{p_1}+\frac{D(r_2)}{p_2}+\frac{F(\pi)}{p_3}$$

and substituting expressions (3.13) in (2.10) and then using

$$s_n=\frac{(n+1/2)\,\pi}{Z_p^r}+\delta_n,$$

we get

$$\delta_n=-\frac{\Delta_p^r}{(n+1/2)\,\pi}+O\left(\frac{1}{n^2}\right)$$

and, finally,

$$s_n=\frac{\left(n+\frac{1}{2}\right)\pi}{Z_p^r}-\frac{\Delta_p^r}{\left(n+\frac{1}{2}\right)\pi}+O\left(\frac{1}{n^2}\right). \tag{3.14}$$

Thus, we have proved the following theorem:

Theorem 3.2. *If the conditions (a) and (b) are satisfied, then the positive eigenvalues $\lambda_n=s_n^2$ of problem (1.1)–(1.7) admit the asymptotic representation (3.14) as $n\to\infty$.*

We can now obtain a more accurate asymptotic formula for the eigenfunctions. It follows from (1.11) and (3.12) that

$$w_1(x,\lambda)=\cos\frac{sx}{p_1}\left[1+\frac{A(x)}{sp_1}\right]-\frac{B(x)\sin\frac{sx}{p_1}}{sp_1}+O\left(\frac{1}{s^2}\right). \tag{3.15}$$

Replacing s with s_n and using (3.14), we get

$$u_{1n}(x)=\cos\frac{\left(n+\frac{1}{2}\right)\pi x}{p_1Z_p^r}\left[1+\frac{A(x)Z_p^r}{\left(n+\frac{1}{2}\right)\pi p_1}\right]+\left[\frac{x\Delta_p^r}{\left(n+\frac{1}{2}\right)\pi p_1}\right]\sin\frac{\left(n+\frac{1}{2}\right)\pi x}{p_1Z_p^r}+O\left(\frac{1}{n^2}\right). \tag{3.16}$$

From (1.12), (2.5), (3.12), (3.13), and (3.15), we obtain

$$w_2(x,\lambda) = \frac{\gamma_1}{\delta_1}\left\{\left[1+\frac{1}{s}\left(\frac{A(r_1)}{p_1}+\frac{C(x)}{p_2}\right)\right]\cos\left(\frac{s}{p_2}\left(\frac{r_1(p_2-p_1)}{2p_1}+x\right)\right)\right.$$

$$\left. -\frac{(D(x)/p_2+B(r_1)/p_1)}{s}\sin\frac{s}{p_2}\left(\frac{r_1(p_2-p_1)}{2p_1}+x\right)\right\}+O\left(\frac{1}{s^2}\right). \qquad (3.17)$$

Further, replacing s with s_n and using (3.14), we find

$$u_{2n}(x) = \frac{\gamma_1}{\delta_1}\left\{\left[1+\frac{Z_p^r\left(\frac{A(r_1)}{p_1}+\frac{C(x)}{p_2}\right)}{\left(n+\frac{1}{2}\right)\pi}\right]\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi}{Z_p^r p_2}\left(\frac{r_1(p_2-p_1)}{2p_1}+x\right)\right)\right.$$

$$+\frac{Z_p^r\Delta_p^r\left(\frac{D(x)}{p_2}+\frac{B(r_1)}{p_1}\right)\left(\frac{r_1(p_2-p_1)}{2p_1}+x\right)}{p_2\left(n+\frac{1}{2}\right)^2\pi^2}$$

$$\left.\times\sin\left(\frac{\left(n+\frac{1}{2}\right)\pi}{Z_p^r p_2}\left(\frac{r_1(p_2-p_1)}{2p_1}+x\right)\right)\right\}+O\left(\frac{1}{n^2}\right). \qquad (3.18)$$

It follows from (1.13), (2.7), (3.12), (3.13), and (3.17) that

$$w_3(x,\lambda) = \frac{\theta_1\gamma_1}{\eta_1\delta_1}\left\{\left[1+\frac{\left(\frac{A(r_1)}{p_1}+\frac{C(r_2)}{p_2}+\frac{E(x)}{p_3}\right)}{s}\right]\right.$$

$$\times\cos\left(\frac{s}{p_3}\left(\frac{p_3(r_1(p_2-p_1)+p_1r_2)-r_2p_1p_2}{p_1p_2}+x\right)\right)$$

$$-\frac{1}{s}\left(\frac{B(r_1)}{p_1}+\frac{D(r_2)}{p_2}+\frac{F(x)}{p_3}\right)$$

$$\left.\times\sin\left(\frac{s}{p_3}\left(\frac{p_3(r_1(p_2-p_1)+p_1r_2)-r_2p_1p_2}{p_1p_2}+x\right)\right)\right\}+O\left(\frac{1}{s^2}\right).$$

Finally, replacing s with s_n and using (3.14), we obtain

$$u_{3n}(x) = \frac{\theta_1\gamma_1}{\eta_1\delta_1}\left\{\left[1 + \frac{Z_p^r\left(\dfrac{A(r_1)}{p_1} + \dfrac{C(r_2)}{p_2} + \dfrac{E(x)}{p_3}\right)}{\left(n+\dfrac{1}{2}\right)\pi}\right]\right.$$

$$\times\cos\left(\frac{\left(n+\dfrac{1}{2}\right)\pi}{Z_p^r p_3}\left(\frac{p_3\left(r_1\left(p_2-p_1\right)+p_1r_2\right)-r_2p_1p_2}{p_1p_2}+x\right)\right)$$

$$+\frac{Z_p^r\Delta_p^r\left(\dfrac{B(r_1)}{p_1} + \dfrac{D(r_2)}{p_2} + \dfrac{F(x)}{p_3}\right)}{p_3\left(n+\dfrac{1}{2}\right)^2\pi^2}\left(\frac{p_3\left(r_1\left(p_2-p_1\right)+p_1r_2\right)-r_2p_1p_2}{p_1p_2}+x\right)$$

$$\left.\times\sin\left(\frac{\left(n+\dfrac{1}{2}\right)\pi}{Z_p^r p_3}\left(\frac{p_3\left(r_1\left(p_2-p_1\right)+p_1r_2\right)-r_2p_1p_2}{p_1p_2}+x\right)\right)\right\} + O\left(\frac{1}{n^2}\right). \tag{3.19}$$

Thus, we have proved the following theorem.

Theorem 3.3. *If the conditions (a) and (b) are satisfied, then the eigenfunctions $u_n(x)$ of problem (1.1)–(1.7) admit the following asymptotic representation for $n \to \infty$:*

$$u_n(x) = \begin{cases} u_{1n}(x), & x \in [0, r_1), \\ u_{2n}(x), & x \in (r_1, r_2), \\ u_{3n}(x), & x \in (r_2, \pi], \end{cases}$$

where $u_{1n}(x),\ u_{2n}(x)$ and $u_{3n}(x)$ are defined as in (3.16), (3.18), and (3.19), respectively.

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