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# Pointwise slant submersions from cosymplectic manifolds

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**Abstract:** In this paper, we characterize the pointwise slant submersions from cosymplectic manifolds onto Riemannian manifolds and give several examples.

**Key words:** Riemannian submersion, almost contact metric manifold, cosymplectic manifold, pointwise slant submersion

## 1. Introduction

An important topic in differential geometry is the Riemannian submersions between Riemannian manifolds introduced by O'Neill [12] and Gray [5]. Such submersions were generalized by Watson to almost Hermitian manifolds by proving that the base manifold and each fiber have the same kind of structure as the total space in most cases [21]. Recently, many works considering different conditions on Riemannian submersion have been done (see [3, 4, 6, 7, 8, 13, 15, 17, 19, 20]).

Sahin [18] introduced slant submersions from almost Hermitian manifolds onto Riemannian manifolds in such a way that let $\pi$ be a Riemannian submersion from an almost Hermitian manifold $(M_1, g_1, J_1)$ onto a Riemannian manifold $(M_2, g_2)$. If for any nonzero vector $X \in \Gamma(\ker \pi_*)$, the angle $\theta(X)$ between $JX$ and the space $\ker \pi_*$ is a constant, i.e. it is independent of the choice of the point $p \in M_1$ and choice of the tangent vector $X$ in $\ker \pi_*$, then we say that $\pi$ is a slant submersion. In this case, the angle $\theta$ is called the slant angle of the slant submersion. He gave some examples and investigated the geometry of leaves of the distributions for such submersions. Furthermore, Lee and Sahin [9] investigated pointwise slant submersions from almost Hermitian manifolds.

In this paper, we define pointwise slant submersions from almost contact metric manifolds onto Riemannian manifolds and give some examples for such submersions. We investigate the harmonicity of pointwise slant submersions from cosymplectic manifolds and obtain necessary and sufficient conditions for the maps to have totally geodesic fibers according to the state being vertical or horizontal of characteristic vector fields. We also give necessary and sufficient conditions for such submersions to be totally geodesic according to the state to be vertical or horizontal of characteristic vector fields.

## 2. Preliminaries

In this section, we provide a brief view of Riemannian submersions and almost contact metric manifolds.

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**2.1. Riemannian submersions**

Let $(M, g_M)$ be an $m$-dimensional Riemannian manifold and $(N, g_N)$ an $n$-dimensional Riemannian manifold. A Riemannian submersion $\pi : M \to N$ is a map of $M$ onto $N$ satisfying the following conditions:

1. $\pi$ has the maximal rank
2. The differential $\pi_*$ preserves the lengths of horizontal vectors.

For each $q \in N$, $\pi^{-1}(q)$ is an $(m-n)$-dimensional submanifold of $M$, so-called fiber. If a vector field on $M$ is always tangent (resp. orthogonal) to fibers, then it is called vertical (resp. horizontal) [12]. A vector field $X$ on $M$ is said to be basic if it is horizontal and $\pi$-related to a vector field $X_*$ on $N$, i.e. $\pi_* X_p = X_{*\pi(p)}$ for all $p \in M$. We denote the projection morphisms on the distributions $\ker \pi_*$ and $(\ker \pi_*)^{\perp}$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

A Riemannian submersion is characterized via O'Neill tensors $\mathcal{T}$ and $\mathcal{A}$ by the formulae

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.1}$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{2.2}$$

for arbitrary vector fields $E$ and $F$ on $M$, where $\nabla$ is the Levi-Civita connection of $(M, g_M)$.

We recall from [12] the following lemma, which will be useful later.

**Lemma 1** *Let $\pi : M \to N$ be a Riemannian submersion between Riemannian manifolds. If $X$ and $Y$ are basic vector fields of $M$, then*

*i)* $g_M(X, Y) = g_N(X_*, Y_*) \circ \pi$,

*ii) the horizontal part $[X, Y]^{\mathcal{H}}$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$ i.e.* $\pi_*\left([X, Y]^{\mathcal{H}}\right) = [X_*, Y_*]$,

*iii)* $[V, X]$ *is vertical for any vector field $V$ of* $\ker \pi_*$,

*iv)* $\left(\nabla^M_X Y\right)^{\mathcal{H}}$ *is the basic vector field corresponding to* $\nabla^N_{X_*} Y_*$,

*where $\nabla^M$ and $\nabla^N$ are the Levi-Civita connection on $M$ and $N$, respectively.*

From (2.1) and (2.2), we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W \tag{2.3}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X \tag{2.4}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V \tag{2.5}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{2.6}$$

for $X, Y \in \Gamma((\ker \pi_*)^{\perp})$ and $V, W \in \Gamma(\ker \pi_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. Moreover, if $X$ is basic, then $\mathcal{H}\nabla_V X = \mathcal{A}_X V$. On the other hand, for any $E \in \Gamma(TM)$, it is seen that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$. The tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy the equations

$$\mathcal{T}_U W = \mathcal{T}_W U, \tag{2.7}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y] \tag{2.8}$$

for $U, W \in \Gamma(\ker \pi_*)$ and $X, Y \in \Gamma\left((\ker \pi_*)^{\perp}\right)$. It can be easily seen that a Riemannian submersion $\pi : M \to N$ has totally geodesic fibers if and only if $\mathcal{T}$ identically vanishes.

Now we consider the notion of harmonic maps between Riemannian manifolds. Let $(M, g_M)$ and $(N, g_N)$ be Riemannian manifolds and suppose that $\psi : M \to N$ is a smooth mapping between them. Then the differential $\psi_*$ of $\psi$ can be considered as a section of the bundle $Hom(TM, \psi^{-1}TN) \to M$, where $\psi^{-1}TN$ is the pullback bundle that has fibers $\left(\psi^{-1}TN\right)_p = T_{\psi(p)}N$, $p \in M$. $Hom(TM, \psi^{-1}TN)$ has a connection $\nabla$ induced from the Levi-Civita connection $\nabla^M$. Then the second fundamental form of $\psi$ is given by

$$\nabla\psi_*(X, Y) = \nabla^{\psi}_X \psi_*(Y) - \psi_*(\nabla^M_X Y) \tag{2.9}$$

for $X, Y \in \Gamma(TM)$, where $\nabla^{\psi}$ is the pullback connection. For a Riemannian submersion $\psi$, we can easily view

$$(\nabla\psi_*)(X, Y) = 0 \tag{2.10}$$

for $X, Y \in \Gamma\left((\ker \psi_*)^{\perp}\right)$. A smooth map $\psi : (M, g_M) \to (N, g_N)$ is said to be harmonic if $trace\nabla\psi_* = 0$ [1].

### 2.2. Almost contact metric manifolds

Let $M$ be a $(2m+1)$-dimensional differentiable manifold with a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$, and a 1-form $\eta$ such that

$$\phi^2 = -I + \eta \otimes \xi, \ \phi\xi = 0, \ \eta \circ \phi = 0, \ \eta(\xi) = 1. \tag{2.11}$$

It is said that $(\phi, \xi, \eta)$ and $(M, \phi, \xi, \eta)$ are respectively called an almost contact structure and almost contact manifold. If there is a Riemannian metric $g$ on almost contact manifold $M$ such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi) \tag{2.12}$$

for any vector fields $X, Y \in \Gamma(TM)$, $(\phi, \xi, \eta, g)$ is called an almost contact metric structure and $(M, \phi, \xi, \eta, g)$ an almost contact metric manifold. The almost contact metric structure $(\phi, \xi, \eta, g)$ is said to be normal if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0, \tag{2.13}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$. *The fundamental* 2*-form* $\Phi$ on $M$ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields $X, Y \in \Gamma(TM)$. An almost contact metric manifold $(M, \phi, \xi, \eta, g)$ is called a cosymplectic manifold if it has a normal almost contact metric structure and both $\Phi$ and $\eta$ are closed, i.e. $d\Phi = 0$ and $d\eta = 0$. Then the structure equation of a cosymplectic manifold $(M, \phi, \xi, \eta, g)$ is given by

$$(\nabla_E \phi) G = 0 \tag{2.14}$$

for any vector fields $E, G \in \Gamma(TM)$, where $\nabla$ is the Levi-Civita connection of the metric $g$ on $M$. Moreover, for a cosymplectic manifold, we have

$$\nabla_E \xi = 0 \tag{2.15}$$

(see [2, 10]).

Now we give some examples for almost contact metric manifolds.

**Example 1** *([14]) We define complex structures $J_1$ and $J_2$ on $\mathbb{R}^4$ as follows:*

$$J_1\left(\frac{\partial}{\partial x_1}\right)=\frac{\partial}{\partial x_2}, J_1\left(\frac{\partial}{\partial x_2}\right)=-\frac{\partial}{\partial x_1}, J_1\left(\frac{\partial}{\partial x_3}\right)=\frac{\partial}{\partial x_4}, J_1\left(\frac{\partial}{\partial x_4}\right)=-\frac{\partial}{\partial x_3},$$
$$J_2\left(\frac{\partial}{\partial x_1}\right)=\frac{\partial}{\partial x_3}, J_2\left(\frac{\partial}{\partial x_2}\right)=-\frac{\partial}{\partial x_4}, J_2\left(\frac{\partial}{\partial x_3}\right)=-\frac{\partial}{\partial x_1}, J_2\left(\frac{\partial}{\partial x_4}\right)=\frac{\partial}{\partial x_2}.$$

*Then $(\phi,\xi,\eta,g)$ is an almost contact metric structure on an Euclidean space $\mathbb{R}^5=\mathbb{R}^4\times\mathbb{R}$ with coordinates $(x_1,x_2,x_3,x_4,x_5)$ such that*

$$\phi\left(X+h\frac{d}{dx_5}\right)=\cos\theta J_1X-\sin\theta J_2X,\ \xi=\frac{d}{dx_5},\ \eta=dx_5$$

*for $X\in\Gamma(T\mathbb{R}^4)$ and $h\in C^\infty(\mathbb{R}^5)$, where $g$ is the Euclidean metric on $\mathbb{R}^5$ and $\theta:\mathbb{R}^5\to[0,\frac{\pi}{2}]$ is a differentiable function.*

**Example 2** *([11]). Let $(x_i,y_i,z)$ be Cartesian coordinates on $\mathbb{R}^{2n+1}$ for $i=1,...,n$. An almost contact metric structure $(\phi,\xi,\eta,g)$ on $\mathbb{R}^{2n+1}$ is defined as follows:*

$$g=\sum_{i=1}^{n}\left((dx_i)^2+(dy_i)^2\right)+(dz)^2,\quad \phi=\begin{pmatrix}0&\delta_{ij}&0\\-\delta_{ij}&0&0\\0&0&0\end{pmatrix},$$

$$\xi=\frac{\partial}{\partial z},\quad \eta=dz$$

*We can easily show that $(\phi,\xi,\eta,g)$ is cosymplectic structure in $\mathbb{R}^{2n+1}$. The vector fields $E_i=\frac{\partial}{\partial y_i}$, $E_{n+i}=\frac{\partial}{\partial x_i}$, $\xi$ form a $\phi$-basis for the cosymplectic structure in $\mathbb{R}^{2n+1}$. Then $\mathbb{R}^{2n+1}$ with the cosymplectic structure $(\phi,\xi,\eta,g)$ is a cosymplectic manifold.*

## 3. The pointwise slant submersions in almost contact manifolds

This section is devoted to defining the pointwise slant submersion from an almost contact metric manifold $(M,\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N,g_N)$ to provide some examples for such submersions.

**Definition 1** *Let $\pi$ be a Riemannian submersion from an almost contact metric manifold $(M,\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N,g_N)$. If for each $p\in M$ the angle $\theta(X)$ between $\phi X$ and the space $(\ker\pi_*)_p$ is independent of the choice of the nonzero vector $X\in\Gamma\left((\ker\pi_*)-\{\xi\}\right)$, then $\pi$ is called a pointwise slant submersion. The angle $\theta$ is called the slant function of the pointwise slant submersion.*

A pointwise slant submersion is called slant if its slant function $\theta$ is independent of the choice of the point on $(M,\phi,\xi,\eta,g_M)$ [9]. Then, the constant $\theta$ is called the *slant angle* of the slant submersion.

We now give the following examples for pointwise slant submersion.

**Example 3** *Let* $\mathbb{R}^5$ *be an almost contact metric manifold as in Example 1. Define a map* $\pi : \mathbb{R}^5 \to \mathbb{R}^2$ *by* $\pi(x_1, x_2, x_3, x_4, x_5) = \left(\frac{1}{\sqrt{2}}(x_1 + x_3), x_4\right)$. *Then*

$$\ker \pi_* = span\left\{V_1 = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right), V_2 = \frac{\partial}{\partial x_2}, V_3 = \xi = \frac{\partial}{\partial x_5}\right\}$$

*and*

$$(\ker \pi_*)^{\perp} = span\left\{H_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right), H_2 = \frac{\partial}{\partial x_4}\right\}.$$

*We can easily show that* $\pi$ *is a Riemannian submersion and* $|g(\phi V_1, V_2)| = \frac{\cos\theta}{\sqrt{2}}$ *on* $\mathbb{R}^5$. *Then* $\pi$ *is a pointwise slant submersion with the slant function* $\theta$.

**Example 4** *Let* $\phi_1$ *and* $\phi_2$ *be two tensor fields on* $\mathbb{R}^5$.

$$\phi_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \phi_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

*Note that* $\phi_1\phi_2 = -\phi_2\phi_1$. *For any real valued function* $\theta : \mathbb{R}^5 \to \mathbb{R}$, *we can denote a new tensor field on* $\mathbb{R}^5$ *by* $\phi = \cos\theta\phi_1 + \sin\theta\phi_2$. *Then* $(\phi, \xi, \eta, g)$ *is an almost contact metric structure on* $\mathbb{R}^5$ *such that* $\eta = dx_5$, $\xi = \frac{\partial}{\partial x_5}$ *and* $g$ *is a Euclidean metric on* $\mathbb{R}^5$. *Define a map* $\pi : \mathbb{R}^5 \to \mathbb{R}^2$ *by* $\pi(x_1, x_2, x_3, x_4, x_5) = \left(\frac{x_1 - x_3}{2}, \frac{x_2 - x_4}{2}\right)$. *Then*

$$\ker \pi_* = span\left\{V_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right), V_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\right), V_3 = \xi = \frac{\partial}{\partial x_5}\right\}$$

*and*

$$(\ker \pi_*)^{\perp} = span\left\{H_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\right), H_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4}\right)\right\}.$$

*We can easily show that* $\pi$ *is a Riemannian submersion and*

$$|g(\phi V_1, V_2)| = |\cos\theta|$$

*on* $\mathbb{R}^5$. *Then* $\pi$ *is a pointwise slant submersion with slant function* $\theta$.

Now we investigate basic properties and derive some theorems depending on characteristic vector field $\xi$ for pointwise slant submersions from cosymplectic manifolds onto Riemannian manifolds.

### 3.1. Pointwise slant submersions for $\xi \in \Gamma(\ker \pi_*)$

Let $\pi$ be a pointwise slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then for $U, V \in \Gamma(\ker \pi_*)$, we have

$$\phi U = \varphi U + \omega U, \tag{3.1}$$

where $\varphi U$ and $\omega U$ are vertical and horizontal parts of $\phi U$, respectively. Moreover, for $X \in \Gamma((\ker \pi_*)^{\perp})$, we have

$$\phi X = \mathcal{B}X + \mathcal{C}X, \tag{3.2}$$

where $\mathcal{B}X$ and $\mathcal{C}X$ are vertical and horizontal components of $\phi X$, respectively. Then, by using (2.12), (3.1), and (3.2), we get

$$g_M(\varphi U, V) = -g_M(U, \varphi V) \tag{3.3}$$

and

$$g_M(\omega U, Y) = -g_M(U, \mathcal{B}Y) \tag{3.4}$$

for $U, V \in \Gamma(\ker \pi_*)$ and $Y \in \Gamma\left((\ker \pi_*)^{\perp}\right)$.

Further, from (2.3) and (2.15), we obtain

$$\mathcal{T}_U \xi = 0 \tag{3.5}$$

for $U \in \Gamma(\ker \pi_*)$.

On the other hand, using (2.3), (2.4), (3.1), and (3.2), we have

$$(\nabla_U \omega) V = \mathcal{C}\mathcal{T}_U V - \mathcal{T}_U \varphi V, \tag{3.6}$$

$$(\nabla_U \varphi) V = \mathcal{B}\mathcal{T}_U V - \mathcal{T}_U \omega V, \tag{3.7}$$

where

$$(\nabla_U \omega) V = \mathcal{H}\nabla_U \omega V - \omega \hat{\nabla}_U V, \tag{3.8}$$

$$(\nabla_U \varphi) V = \hat{\nabla}_U \varphi V - \varphi \hat{\nabla}_U V \tag{3.9}$$

for $U, V \in \Gamma(\ker \pi_*)$, where $\nabla$ is the Levi-Civita connection on $M$. We say that $\omega$ is parallel if

$$(\nabla_U \omega) V = 0$$

for $U, V \in \Gamma(\ker \pi_*)$.

We now give a characterization theorem for pointwise slant submersions.

**Theorem 1** *Let $\pi$ be a Riemannian submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then $\pi$ is a pointwise slant submersion if and only if*

$$\varphi^2 = \cos^2\theta(-I + \eta \otimes \xi),$$

*where $\theta$ is a real-valued function.*

**Proof** From Definition 1 we can write

$$\cos\theta = \frac{g_M(\phi U, \varphi U)}{|\phi U|\,|\varphi U|} = \frac{|\varphi U|}{|\phi U|} \tag{3.10}$$

for $U \in \Gamma(\ker \pi_*)$. If we write $\varphi U$ instead of $U$ in (3.10) and using (2.12) we get

$$\cos\theta = \frac{|\varphi^2 U|}{|\varphi U|}. \tag{3.11}$$

Furthermore, from (3.3) we obtain

$$g_M(\varphi^2 U, U) = -g_M(\varphi U, \varphi U) = -\left|\varphi U\right|^2. \tag{3.12}$$

Using (3.10), (3.11), and (3.12), we have

$$g_M\left(\varphi^2 U, \phi^2 U\right) = \left|\varphi^2 U\right| \left|\phi^2 U\right|. \tag{3.13}$$

Then from (2.11), (3.10), (3.11), and (3.13), we arrive at

$$\varphi^2 = \cos^2\theta(-I + \eta \otimes \xi).$$

for any $U \in \Gamma(\ker \pi_*)$.

Conversely, it can be directly verified. □

Using (2.11), (3.1), and Theorem 1, we have the following Lemma.

**Lemma 2** *Let $\pi$ be a pointwise slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$ with slant function $\theta$. Then the following equations are satisfied:*

$$\begin{aligned} g_M(\varphi U, \varphi V) &= \cos^2\theta \left(g_M(U, V) - \eta(U)\eta(V)\right), \\ g_M(\omega U, \omega V) &= sin^2\theta \left(g_M(U, V) - \eta(U)\eta(V)\right) \end{aligned}$$

*for any $U, V \in \Gamma(\ker \pi_*)$.*

**Lemma 3** *Let $\pi$ be a pointwise slant submersion from cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto Riemannian manifold $(N, g_N)$ with slant function $\theta$. If $\omega$ is parallel then we have*

$$\mathcal{T}_{\varphi U}\varphi U = -\cos^2\theta \mathcal{T}_U U.$$

**Proof** Assume that $\omega$ is parallel. Then using (2.7) from (3.6) we get

$$\mathcal{T}_V \varphi U = \mathcal{C}\mathcal{T}_U V \tag{3.14}$$

for $U, V \in \Gamma(\ker \pi_*)$. By replacing $U$ and $V$ and using (2.7) we have

$$\mathcal{T}_U \varphi V = \mathcal{T}_V \varphi U. \tag{3.15}$$

If we write $\varphi U$ instead of $V$ in (3.15), then the proof is completed. □

Now we examine some properties for pointwise slant submersions from cosymplectic manifolds onto Riemannian manifolds.

If $\pi$ is a map between Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$, then the adjoint map ${}^*\pi_*$ of $\pi_*$ is characterized by

$$g_1(X, {}^*\pi_{*p} Y) = g_2(\pi_{*p} X, Y) \tag{3.16}$$

for $X \in T_p M_1, Y \in T_{\pi(p)} M_2$ and $p \in M_1$. For each $p \in M_1$, $\pi_*^h$ is a map defined by

$$\pi_{*p}^h : \left(\left(\ker \pi_*\right)^{\perp}(p), g_{1p((\ker \pi_*)^{\perp})(p)}\right) \to \left(range\pi_*(q), g_{2q(range\pi_*)(q)}\right),$$

Denote the adjoint of $\pi_*^h$ by ${}^*\pi_{*p}^h$. Let ${}^*\pi_{*p}$ be the adjoint of $\pi_{*p}$ that is defined by

$$\pi_{*p} : (T_pM_1, g_{1p}) \to (T_qM_2, g_{2q}).$$

Then the linear transformation

$$({}^*\pi_{*p})^h : range\pi_*(q) \to (\ker\pi_*)^{\perp}(p)$$

defined as $({}^*\pi_{*p})^h Y =^* \pi_{*p}Y$, where $Y \in \Gamma(range\pi_{*p})$, $q = \pi(p)$, is an isomorphism and $\left(\pi_{*p}^h\right)^{-1} = ({}^*\pi_{*p})^h =^* \left(\pi_{*p}^h\right)$ [1].

**Theorem 2** *Let $\pi$ be a pointwise slant submersion from cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$ with nonzero slant function $\theta$. Then $\pi$ is harmonic if and only if*

$$trace^*\pi_*\left(\left(\nabla\pi_*\right)\left((.)\,\omega\varphi\,(.)\right)\right) - trace\omega\mathcal{T}_{(.)}\omega(.) + trace\mathcal{C}^*\pi_*\left(\nabla\pi_*\right)\left((.), \omega(.)\right) = 0.$$

**Proof** By using the equations (2.3), (2.11), (2.12), (2.14), and (3.1) we obtain

$$\begin{aligned} g_M\left(\mathcal{T}_U U, X\right) &= g_M\left(\nabla_U \varphi U, \phi X\right) + g_M\left(\nabla_U \omega U, \phi X\right) \\ &= -g_M\left(\nabla_U \phi\varphi U, X\right) + g_M\left(\nabla_U \omega U, \phi X\right) \end{aligned}$$

for $U \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$. From (2.4), (3.1), (3.2), and Theorem 1, we deduce

$$\begin{aligned} g_M\left(\mathcal{T}_U U, X\right) &= -\sin 2\theta U(\theta)\left[g_M\left(U, X\right) - g_M\left(\xi, X\right)\right] \\ &+ cos^2\theta\left[g_M\left(\nabla_U U, X\right) - \nabla_U \eta(U) g_M(\xi, X)\right] \\ &- g_M\left(\nabla_U \omega\varphi U, X\right) + g_M\left(\mathcal{T}_U \omega U, \mathcal{B}X\right) \\ &+ g_M\left(\nabla_U \omega U, \mathcal{C}X\right). \end{aligned}$$

Since $\xi \in \Gamma(\ker\pi_*)$, we have

$$\begin{aligned} \sin^2\theta g_M\left(\mathcal{T}_U U, X\right) &= -g_M\left(\nabla_U \omega\varphi U, X\right) + g_M\left(\mathcal{T}_U \omega U, \mathcal{B}X\right) \\ &+ g_M\left(\nabla_U \omega U, \mathcal{C}X\right). \end{aligned}$$

Then using (2.11) and (2.12) we derive

$$\begin{aligned} \sin^2\theta g_M\left(\mathcal{T}_U U, X\right) &= -g_M\left(\nabla_U \omega\varphi U, X\right) - g_M\left(\phi\left(\mathcal{T}_U \omega U\right), X\right) \\ &+ g_M\left(\nabla_U \omega U, \mathcal{C}X\right). \end{aligned}$$

Thus from (2.9) and (3.1) we get

$$\begin{aligned} \sin^2\theta g_M\left(\mathcal{T}_U U, X\right) &= g_N\left(\left(\nabla\pi_*\right)\left(U, \omega\varphi U\right), \pi_*(X)\right) - g_M\left(\omega\mathcal{T}_U \omega U, X\right) \\ &- g_N\left(\left(\nabla\pi_*\right)\left(U, \omega U\right), \pi_*\left(\mathcal{C}X\right)\right) \end{aligned}$$

By (3.16) in the last equation, we derive

$$\begin{aligned} \sin^2\theta g_M\left(\mathcal{T}_U U, X\right) &= g_M\left({}^*\pi_*\left(\left(\nabla\pi_*\right)\left(U, \omega\varphi U\right)\right), X\right) - g_M\left(\omega\mathcal{T}_U \omega U, X\right) \\ &+ g_M\left(\mathcal{C}^*\pi_*\left(\left(\nabla\pi_*\right)\left(U, \omega U\right)\right), X\right). \end{aligned}$$

Conversely, a direct computation gives the proof. □

**Theorem 3** *Let $\pi$ be a pointwise slant submersion from cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then the fibers are totally geodesic submanifolds in $M$ if and only if*

$$\begin{aligned} g_N\left(\nabla^N_{X'}\pi_*(\omega U), \pi_*(\omega V)\right) &= -\sin^2\theta g_M([U,X],V) + \sin 2\theta X(\theta) g_M(\phi U, \phi V) \\ &+ g_M(\mathcal{A}_X\omega\varphi U, V) - g_M(\mathcal{A}_X\omega U, \varphi V) \\ &- \sin^2\theta\eta(\nabla_X U)\eta(V), \end{aligned}$$

*where $X$ and $X'$ are $\pi$-related vector fields and $\nabla^N$ is the Levi-Civita connection on $N$.*

**Proof** By using (2.4), (2.11), (2.12), (2.14), (3.1), and (3.2), we get

$$\begin{aligned} g_M(\mathcal{T}_U V, X) &= -g_M([U,X],V) + g_M\left(\nabla_X\varphi^2 U, V\right) + g_M(\nabla_X\omega\varphi U, V) \\ &- g_M(\nabla_X\omega U, \phi V) - \eta(\nabla_X U)\eta(V) \end{aligned}$$

for $U, V \in \Gamma(\ker\pi_*)$ and $X \in \Gamma\left((\ker\pi_*)^\perp\right)$. Then it follows from Theorem 1, (2.6), and (2.10) that

$$\begin{aligned} g_M(\mathcal{T}_U V, X) &= -g_M([U,X],V) + \sin 2\theta X(\theta) g_M(U,V) - \cos^2\theta g_M(\nabla_X U, V) \\ &+ g_M(\mathcal{A}_X\omega\varphi U, V) - g_M(\mathcal{A}_X\omega U, \varphi V) \\ &- g_N\left(\nabla^N_{X'}\pi_*(\omega U), \pi_*(\omega V)\right) - \sin 2\theta X(\theta)\eta(U)\eta(V) \\ &- \sin^2\theta\eta(\nabla_X U)\eta(V). \end{aligned}$$

Since $\mathcal{T}$ is skew symmetric and from (2.3) we conclude

$$\begin{aligned} \sin^2\theta g_M(\mathcal{T}_U V, X) &= -\sin^2\theta g_M([U,X],V) + \sin 2\theta X(\theta) g_M(\phi U, \phi V) \\ &+ g_M(\mathcal{A}_X\omega\varphi U, V) - g_M(\mathcal{A}_X\omega U, \varphi V) \\ &- g_N\left(\nabla^N_{X'}\pi_*(\omega U), \pi_*(\omega V)\right) - \sin^2\theta\eta(\nabla_X U)\eta(V). \end{aligned}$$

By considering the fibers as totally geodesic, we derive the formula given in the hypothesis.
Conversely, it can be directly verified. □

**Theorem 4** *Let $\pi$ be a pointwise slant submersion from cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then $\pi$ is a totally geodesic map if and only if*

$$\begin{aligned} g_N\left(\nabla^N_{X'}\pi_*(\omega U), \pi_*(\omega V)\right) &= -\sin^2\theta g_M([U,X],V) + \sin 2\theta X(\theta) g_M(\phi U, \phi V) \\ &+ g_M(\mathcal{A}_X\omega\varphi U, V) - g_M(\mathcal{A}_X\omega U, \varphi V) \\ &- \sin^2\theta\eta(\nabla_X U)\eta(V), \end{aligned}$$

*and*

$$g_M(\mathcal{A}_X\omega U, \mathcal{B}Y) = g_N(\nabla^\pi_X\pi_*(\omega\varphi U), \pi_*(Y)) - g_N(\nabla^\pi_{X'}\pi_*(\omega U), \pi_*(\mathcal{C}Y))$$

*where $X$ and $X'$ are $\pi$-related vector fields and $\nabla^\pi$ is the pull-back connection along $\pi$.*

**Proof** By definition, it follows that $\pi$ is totally geodesic if and only if

$$(\nabla\pi_*)(X,Y)=0$$

for any $X,Y\in\Gamma(TM)$. Using Theorem 3 and (2.10), we obtain the first equation. On the other hand, from Theorem 1, (2.6), (2.9), (2.11), (2.12), (2.14), (3.1), and (3.2), we have

$$\begin{aligned} g_N\left(\left(\nabla^N\pi_*\right)(X,U),\pi_*(Y)\right) &= -\sin 2\theta X(\theta)g_M(U,Y)-\cos^2\theta g_M(\nabla_X U,Y)\\ &+ g_M(\nabla_X\omega\varphi U,Y)-g_M(\mathcal{A}_X\omega U,\mathcal{B}Y)\\ &- g_M(\nabla_X\omega U,\mathcal{C}Y)\end{aligned}$$

Then from (2.9) and (2.10) we get

$$\begin{aligned} \sin^2\theta g_N((\nabla\pi_*)(X,U),\pi_*(Y)) &= g_N(\nabla^\pi_X\pi_*(\omega\varphi U),\pi_*(Y))\\ &- g_N(\nabla^\pi_X\pi_*(\omega U),\pi_*(\mathcal{C}Y))\\ &- g_M(\mathcal{A}_X\omega U,\mathcal{B}Y).\end{aligned}$$

Conversely it is proved by direct calculation. □

### 3.2. Pointwise slant submersions for $\xi\in\Gamma\left((\ker\pi_*)^\perp\right)$

In this section, we give the basic equations of pointwise slant submersion from cosymplectic manifolds onto Riemannian manifolds for $\xi\in\Gamma(\ker\pi_*)^\perp$. Using (2.11) and (2.12) we have

$$\phi^2U=-U \tag{3.17}$$

and

$$g(\phi U,\phi V)=g(U,V), \tag{3.18}$$

for $U,V\in\Gamma(\ker\pi_*)$. Moreover, from (2.3) and (2.15), we derive

$$\mathcal{T}_U\xi=0 \tag{3.19}$$

and by using $g(U,\xi)=0$,

$$\eta(\nabla_U V)=0, \tag{3.20}$$

for $U,V\in\Gamma((\ker\pi_*))$. On the other hand, from (2.6)

$$\mathcal{A}_X\xi=0 \tag{3.21}$$

for any $X\in\Gamma\left((\ker\pi_*)^\perp\right)$.

The following theorems can be proved by the arguments used in the previous section.

**Theorem 5** *Let $\pi$ be a Riemannian submersion from a cosymplectic manifold $(M,\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N,g_N)$. Then $\pi$ is a pointwise slant submersion if and only if*

$$\varphi^2=-\left(\cos^2\theta\right)I,$$

*where $\theta$ is a real-valued function.*

**Lemma 4** *Let* $\pi$ *be a pointwise slant submersion from a cosymplectic manifold* $(M, \phi, \xi, \eta, g_M)$ *onto a Riemannian manifold* $(N, g_N)$ *with slant function* $\theta$. *Then we have the following relations:*

$$g_M(\varphi U, \varphi V) = \cos^2\theta g_M(U, V),$$
$$g_M(\omega U, \omega V) = sin^2\theta g_M(U, V)$$

*for any* $U, V \in \Gamma(\ker \pi_*)$.

**Lemma 5** *Let* $\pi$ *be a pointwise slant submersion from cosymplectic manifold* $(M, \phi, \xi, \eta, g_M)$ *onto Riemannian manifold* $(N, g_N)$ *with slant function* $\theta$. *If* $\omega$ *is parallel, then we have*

$$\mathcal{T}_{\varphi U}\varphi U = -\cos^2\theta \mathcal{T}_U U$$

**Theorem 6** *Let* $\pi$ *be a pointwise slant submersion from cosymplectic manifold* $(M, \phi, \xi, \eta, g_M)$ *onto a Riemannian manifold* $(N, g_N)$. *Then* $\pi$ *is harmonic if and only if*

$$trace^*\pi_*\left((\nabla\pi_*)\left((.)\,\omega\varphi\,(.)\right)\right) - trace\omega\mathcal{T}_{(.)}\omega(.) + trace\mathcal{C}^*\pi_*\,(\nabla\pi_*)\,((.), \omega(.)) = 0.$$

**Theorem 7** *Let* $\pi$ *be a pointwise slant submersion from cosymplectic manifold* $(M, \phi, \xi, \eta, g_M)$ *onto a Riemannian manifold* $(N, g_N)$. *Then the fibers are totally geodesic submanifolds in* $M$ *if and only if*

$$\begin{aligned} g_N\left(\nabla^N_{X'}\pi_*\left(\omega U\right), \pi_*\left(\omega V\right)\right) &= -\sin^2\theta g_M\left(\left[U, X\right], V\right) + \sin 2\theta X(\theta) g_M\left(U, V\right) \\ &+ g_M\left(\mathcal{A}_X\omega\varphi U, V\right) - g_M\left(\mathcal{A}_X\omega U, \varphi V\right), \end{aligned}$$

*where* $X$ *and* $X'$ *are* $\pi$*-related vector fields and* $\nabla^N$ *is the Levi-Civita connection on* $N$.

**Theorem 8** *Let* $\pi$ *be a pointwise slant submersion from cosymplectic manifold* $(M, \phi, \xi, \eta, g_M)$ *onto a Riemannian manifold* $(N, g_N)$. *Then* $\pi$ *is a totally geodesic map if and only if*

$$\begin{aligned} g_N\left(\nabla^N_{X'}\pi_*\left(\omega U\right), \pi_*\left(\omega V\right)\right) &= -\sin^2\theta g_M\left(\left[U, X\right], V\right) + \sin 2\theta X(\theta) g_M\left(U, V\right) \\ &+ g_M\left(\mathcal{A}_X\omega\varphi U, V\right) - g_M\left(\mathcal{A}_X\omega U, \varphi V\right) \end{aligned}$$

*and*

$$g_M\left(\mathcal{A}_X\omega U, \mathcal{B}Y\right) = g_N\left(\nabla^\pi_X\pi_*\left(\omega\varphi U\right), \pi_*\left(Y\right)\right) - g_N\left(\nabla^\pi_{X'}\pi_*\left(\omega U\right), \pi_*\left(\mathcal{C}Y\right)\right),$$

*where* $X$ *and* $X'$ *are* $\pi$*-related vector fields and* $\nabla^\pi$ *is the pull-back connection along* $\pi$.

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