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On the extended Kim's p-adic q-deformed fermionic integrals in the p-adic integer ring

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ABSTRACT

The purpose of this paper is to derive some applications of umbral calculus by using extended fermionic p-adic q-integral on $\mathbb{Z}_p$. From those applications, we derive some new interesting properties on the new family of Euler numbers and polynomials. That is, a systemic study of the class of Sheffer sequences in connection with generating function of the weighted q-Euler polynomials is given in the present paper.

1. Preliminaries

Suppose that p be a fixed odd prime number. Throughout this work we use the following notations, where $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. The p-adic absolute value is defined by $|p|_p = p^{-1}$. Also, we assume that $|q-1|_p < 1$ is an indeterminate. Let $UD(\mathbb{Z}_p)$ be the

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space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-deformed integral on $\mathbb{Z}_p$ is defined by T. Kim, as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(\xi)\,d\mu_{-q}(\xi) = \lim_{n\to\infty} \frac{1}{[p^n]_{-q}} \sum_{\xi=0}^{p^n-1} (-1)^{\xi} f(\xi) q^{\xi}, \tag{1.1}$$

where $[x]_q$ is q-analogue of x defined by

$$[x]_q = \frac{q^x-1}{q-1} \quad \text{and} \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

We want to note that $\lim_{q\to1}[x]_q = x$ (for details, see [1–28]).

By (1.1), we have

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0) \tag{1.2}$$

where $f_1(\xi) := f(\xi+1)$ (for details, see [12,16]).

Let us consider Kim's p-adic q-deformed integral on $\mathbb{Z}_p$ in the following form:

For $|1-\zeta|_p < 1$

$$I_{-q}^{\zeta}(f) = \int_{\mathbb{Z}_p} \zeta^{\xi} f(\xi)\,d\mu_{-q}(\xi) = \lim_{n\to\infty} \frac{1}{[p^n]_{-q}} \sum_{\xi=0}^{p^n-1} \zeta^{\xi} f(\xi)(-1)^{\xi} q^{\xi}, \tag{1.3}$$

where $I_{-q}^{\zeta}(f)$ are called extended fermionic p-adic q-integral on $\mathbb{Z}_p$.

Let us now consider $f_1(\xi) := f(\xi+1)$, then we develop as follows:

$$\begin{aligned} -q\zeta I_{-q}^{\zeta}(f_1) &= \lim_{n\to\infty} \frac{1}{[p^n]_{-q}} \sum_{\xi=0}^{p^n-1} \zeta^{\xi+1}(-1)^{\xi+1} f(\xi+1) q^{\xi+1} \\ &= I_{-q}^{\zeta}(f) + \frac{[2]_q}{2} \lim_{n\to\infty}\left(-f(0) - \xi^{p^n} f\left(p^n\right) q^{p^n}\right) \\ &= I_{-q}^{\zeta}(f) - [2]_q f(0). \end{aligned}$$

Therefore, we have the following lemma.

Lemma 1. *For $f \in UD(\mathbb{Z}_p)$, we get*

$$q\zeta I_{-q}^{\zeta}(f_1) + I_{-q}^{\zeta}(f) = [2]_q f(0).$$

Taking $f(\xi) = e^{t(x+\xi)} \in UD(\mathbb{Z}_p)$ in Lemma 1, then we introduce the following expression:

$$\int_{\mathbb{Z}_p} \zeta^{\xi} e^{t(x+\xi)}\,d\mu_{-q}(\xi) = \frac{[2]_q}{q\zeta e^t + 1} e^{tx} = \sum_{n=0}^{\infty} E_{n,\zeta}^{q}(x) \frac{t^n}{n!}, \tag{1.4}$$

where we call $E_{n,\zeta}^{q}(x)$ as weighted q-Euler polynomials. In the special case, $x=0$, $E_{n,\zeta}^{q}(0) := E_{n,\zeta}^{q}$ are called weighted q-Euler numbers and the relation between weighted q-Euler numbers and weighted q-Euler polynomials is given by

$$E_{n,\zeta}^{q}(x)=\sum_{l=0}^{n}\binom{n}{l}x^{l}E_{n-l,\zeta}^{q}=\left(x+E_{\zeta}^{q}\right)^{n}, \tag{1.5}$$

with the usual of replacing $(E_{\zeta}^{q})^{n}$ by $E_{n,\zeta}^{q}$ is used. By (1.4), we note that

$$E_{n,\zeta}^{q}=\int_{\mathbb{Z}_p}\zeta^{\xi}\xi^{n}\,d\mu_{-q}(\xi)\quad\text{and}\quad E_{n,\zeta}^{q}(x)=\int_{\mathbb{Z}_p}\zeta^{\xi}(x+\xi)^{n}\,d\mu_{-q}(\xi). \tag{1.6}$$

By (1.4), we have

$$E_{n,\zeta}^{q}(x)=[2]_q\sum_{m=0}^{\infty}(-1)^{m}q^{m}\zeta^{m}(m+x)^{n},\quad\text{for } n\in\mathbb{N}^{*}. \tag{1.7}$$

From this, we can define weighted q-Zeta function as follows:

$$\lambda(s,x:q:\zeta)=[2]_q\sum_{m=0}^{\infty}\frac{(-1)^{m}q^{m}\zeta^{m}}{(m+x)^{s}}. \tag{1.8}$$

By (1.7) and (1.8), we derive the following equation (1.9):

$$\lambda(-n,x:q:\zeta)=E_{n,\zeta}^{q}(x),\quad\text{for any } n\in\mathbb{N}^{*}. \tag{1.9}$$

When we set as $q=\zeta=1$ in (1.9) reduces to

$$\zeta_E(-n,x)=E_n(x)$$

which is well known in [20].

By (1.3) and (1.4), we compute

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\zeta^{\xi}(x+\xi)^{n}\,d\mu_{-q}(\xi)\\
&=\lim_{m\to\infty}\frac{1}{[dp^{m}]_{-q}}\sum_{\xi=0}^{dp^{m}-1}(-1)^{\xi}\zeta^{\xi}(x+\xi)^{n}q^{\xi}\\
&=\frac{d^{n}}{[d]_{-q}}\sum_{j=0}^{d-1}(-1)^{j}\zeta^{j}q^{j}\left(\lim_{m\to\infty}\frac{1}{[p^{m}]_{(-q)^{d}}}\sum_{\xi=0}^{p^{m}-1}(-1)^{\xi}\left(\zeta^{d}\right)^{\xi}\left(q^{d}\right)^{\xi}\left(\frac{x+j}{d}+\xi\right)^{n}\right)\\
&=\frac{d^{n}}{[d]_{-q}}\sum_{j=0}^{d-1}(-1)^{j}\zeta^{j}q^{j}\int_{\mathbb{Z}_p}\zeta^{d\xi}\left(\frac{x+j}{d}+\xi\right)^{n}d\mu_{-q^{d}}(\xi),
\end{aligned}$$

where d is an odd natural number. So from the above

$$\int_{\mathbb{Z}_p}\zeta^{\xi}(x+\xi)^{n}\,d\mu_{-q}(\xi)=\frac{d^{n}}{[d]_{-q}}\sum_{j=0}^{d-1}(-1)^{j}\zeta^{j}q^{j}\int_{\mathbb{Z}_p}\zeta^{d\xi}\left(\frac{x+j}{d}+\xi\right)^{n}d\mu_{-q^{d}}(\xi). \tag{1.10}$$

By (1.6) and (1.10), we get

$$E_{n,\zeta}^{q}(dx)=\frac{d^n}{[d]_{-q}}\sum_{j=0}^{d-1}(-1)^j\zeta^j q^j E_{n,\zeta^d}^{q^d}\left(x+\frac{j}{d}\right),\tag{1.11}$$

which plays an important role for studying regarding Measure theory on p-adic analysis.

Let we use the following notations, where $\mathbb{C}$ denotes the set of complex numbers, $\mathcal{F}$ denotes the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\;\middle|\;a_k\in\mathbb{C}\right\},$$

$\mathcal{P}=\mathbb{C}[x]$ and $\mathcal{P}^*$ denotes the vector space of all linear functionals on $\mathcal{P}$, $\langle L\mid p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, and it is well known that the vector space operation on $\mathcal{P}^*$ is defined by

$$\langle L+M\mid p(x)\rangle=\langle L\mid p(x)\rangle+\langle M\mid p(x)\rangle,$$
$$\langle cL\mid p(x)\rangle=c\langle L\mid p(x)\rangle,$$

where c is any constant in $\mathbb{C}$ (for details, see [6,18–24,26,28]).

The formal power series are known by

$$f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\in\mathcal{F}$$

which describes a linear functional on $\mathcal{P}$ as $\langle f(t)\mid x^n\rangle=a_n$ for all $n\geqslant 0$ (for details, see [6,18–24,26,28]). In addition to

$$\langle t^k\mid x^n\rangle=n!\delta_{n,k},\tag{1.12}$$

where $\delta_{n,k}$ is the well-known Kronecker delta that returns 1 iff arguments are equal and 0 otherwise. If we take as

$$f_L(t)=\sum_{k=0}^{\infty}\langle L\mid x^k\rangle\frac{t^k}{k!},$$

then we obtain

$$\langle f_L(t)\mid x^n\rangle=\langle L\mid x^n\rangle$$

and so as linear functionals $L=f_L(t)$ (see [6,18–24,26,28]). Additionally, the map $L\rightarrow f_L(t)$ is a vector space isomorphism from $\mathcal{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of the formal power series in t and the vector space of all linear functionals on $\mathcal{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. $\mathcal{F}$ will be called as umbral algebra (see [6,18–24,26,28]).

It is well known that $\langle e^{yt}\mid x^n\rangle=y^n$. Then, leads to the following

$$\langle e^{yt}\mid p(x)\rangle=p(y)$$

(see [6,18–24,26–28]). We want to note that for all $f(t)$ in $\mathcal{F}$

$$f(t)=\sum_{k=0}^{\infty}\langle f(t)\mid x^k\rangle\frac{t^k}{k!}\tag{1.13}$$

and for all polynomials $p(x)$,

$$p(x)=\sum_{k=0}^{\infty}\langle t^k\mid p(x)\rangle\frac{x^k}{k!}\tag{1.14}$$

(for details, see [6,18–24,26,28]). The order $o(f(t))$ of the power series $f(t)\neq 0$ is the smallest integer k for which a_k does not vanish. It is considered $o(f(t))=\infty$ if $f(t)=0$. We see that $o(f(t)g(t))=o(f(t))+o(g(t))$ and $o(f(t)+g(t))\geqslant \min\{o(f(t)),o(g(t))\}$. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$, if and only if $o(f(t))=0$. Such series is called an invertible series. A series $f(t)$ for which $o(f(t))=1$ is called a delta series (see [6,18–24,26–28]). For $f(t),g(t)\in\mathcal{F}$, we have $\langle f(t)g(t)\mid p(x)\rangle=\langle f(t)\mid g(t)p(x)\rangle$.

A delta series $f(t)$ has a compositional inverse $\overline{f}(t)$ such that $f(\overline{f}(t))=\overline{f}(f(t))=t$.

For $f(t),g(t)\in\mathcal{F}$, we have $\langle f(t)g(t)\mid p(x)\rangle=\langle f(t)\mid g(t)p(x)\rangle$. By (1.13), we have

$$p^{(k)}(x)=\frac{d^kp(x)}{dx^k}=\sum_{l=k}^{\infty}\frac{\langle t^l\mid p(x)\rangle}{l!}l(l-1)\cdots(l-k+1)x^{l-k}.\tag{1.15}$$

Thus, notice that

$$p^{(k)}(0)=\langle t^k\mid p(x)\rangle=\langle 1\mid p^{(k)}(x)\rangle.\tag{1.16}$$

By (1.15), we have

$$t^kp(x)=p^{(k)}(x)=\frac{d^kp(x)}{dx^k}.\tag{1.17}$$

So from the above

$$e^{yt}p(x)=p(x+y).\tag{1.18}$$

Let $S_n(x)$ be a polynomial with $\deg S_n(x)=n$. Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\langle g(t)f(t)^k\mid S_n(x)\rangle=n!\delta_{n,k}$ for all $n,k\geqslant 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t),f(t))$ or that $S_n(t)$ is Sheffer for $(g(t),f(t))$.

The Sheffer sequence for $(1,f(t))$ is called the associated sequence for $f(t)$ or $S_n(x)$ is associated to $f(t)$. The Sheffer sequence for $(g(t),t)$ is called the Appell sequence for $g(t)$ or $S_n(x)$ is Appell for $g(t)$.

Let $p(x)\in\mathcal{P}$. Then we have

$$\begin{aligned}\langle f(t)\mid xp(x)\rangle&=\langle\partial_tf(t)\mid p(x)\rangle=\langle f'(t)\mid p(x)\rangle,\\ \langle e^{yt}+1\mid p(x)\rangle&=p(y)+p(0)\quad(\text{see [28]}).\end{aligned}\tag{1.19}$$

Let $S_n(x)$ be Sheffer for $(g(t), f(t))$. Then

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid S_k(x)\rangle}{k!} g(t) f(t)^k, \quad h(t) \in \mathcal{F},$$

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t) f(t)^k \mid p(x)\rangle}{k!} S_k(x), \quad p(x) \in \mathcal{P},$$

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} S_k(y)\frac{t^k}{k!}, \quad \text{for all } y \in \mathbb{C}, \tag{1.20}$$

$$f(t)S_n(x) = nS_{n-1}(x).$$

Also, it is well known in [28] that

$$\langle f_1(t) f_2(t) \cdots f_m(t) \mid x^n \rangle = \sum \binom{n}{i_1, \ldots, i_m} \langle f_1(t) \mid x^{i_1} \rangle \cdots \langle f_m(t) \mid x^{i_m} \rangle \tag{1.21}$$

where $f_1(t), f_2(t), \ldots, f_m(t) \in \mathcal{F}$ and the sum is over all nonnegative integers $i_1, \ldots, i_m$ such that $i_1 + \cdots + i_m = n$ (see [28]).

In [6], Dere and Simsek have studied applications of umbral algebra to special functions. Kim et al. have given some properties of umbral calculus for Frobenius–Euler polynomials [18], Euler polynomials [24] and other special functions [26]. Also, they investigated some new applications of umbral calculus associated with p-adic invariants integral on $\mathbb{Z}_p$ in [17].

By the same motivation, we also give some applications of umbral calculus by using extended fermionic p-adic q-integral on $\mathbb{Z}_p$. From those applications, we derive some interesting equalities on weighted q-Euler numbers, weighted q-Euler polynomials and weighted q-Euler polynomials of order k.

2. On the extended fermionic p-adic q-integrals on $\mathbb{Z}_p$ in connection with applications of umbral calculus

Suppose that $S_n(x)$ is an Appell sequence for $g(t)$. Then, by (1.20), we have

$$\frac{1}{g(t)} x^n = S_n(x) \quad \text{if and only if} \quad x^n = g(t)S_n(x) \quad (n \geqslant 0). \tag{2.1}$$

Let us contemplate as follows:

$$g_q(t \mid \zeta) = \frac{q\zeta e^t + 1}{[2]_q} \in \mathcal{F}.$$

Therefore, we easily notice that $g(t)$ is an invertible series. By (2.1), we have

$$\sum_{n=0}^{\infty} E_{n,\zeta}^q(x) \frac{t^n}{n!} = \frac{1}{g_q(t \mid \zeta)} e^{xt}. \tag{2.2}$$

By (2.2), we have

$$\frac{1}{g_q(t \mid \zeta)} x^n = E_{n,\zeta}^q(x). \tag{2.3}$$

Also, by (1.20), we have

$$tE^q_{n,\zeta}(x) = \left(E^q_{n,\zeta}(x)\right)' = nE^q_{n-1,\zeta}(x). \tag{2.4}$$

By (2.3) and (2.4), we have the following proposition.

Proposition 1. *For $n \geqslant 0$, $E^q_{n,\zeta}(x)$ is an Appell sequence for $g_q(t \mid \zeta) = \frac{\zeta q e^t + 1}{[2]_q}$.*

By (1.6), we derive that

$$\begin{aligned}\sum_{n=1}^{\infty} E^q_{n,\zeta}(x)\frac{t^n}{n!} &= \frac{x g_q(t \mid \zeta)e^{xt} - g'_q(t \mid \zeta)e^{xt}}{g(t)^2} \\ &= \sum_{n=0}^{\infty}\left(x\frac{1}{g_q(t \mid \zeta)}x^n - \frac{g'_q(t \mid \zeta)}{g_q(t \mid \zeta)}\frac{1}{g_q(t \mid \zeta)}x^n\right)\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

Because of (2.3) and (2.5), we discover the following:

$$E^q_{n+1,\zeta}(x) = xE^q_{n,\zeta}(x) - \frac{g'_q(t \mid \zeta)}{g_q(t \mid \zeta)}E^q_{n,\zeta}(x).$$

Therefore, we get the following theorem.

Theorem 1. *Let $g_q(t \mid \zeta) = \frac{\zeta q e^t + 1}{2} \in \mathcal{F}$. Then we have for $n \geqslant 0$:*

$$E^q_{n+1,\zeta}(x) = \left(x - \frac{g'_q(t \mid \zeta)}{g_q(t \mid \zeta)}\right)E^q_{n,\zeta}(x). \tag{2.6}$$

Moreover,

$$\lambda(-n-1, x : q : \zeta) = \left(x - \frac{g'_q(t \mid \zeta)}{g_q(t \mid \zeta)}\right)\lambda(-n, x : q : \zeta),$$

where $g'_q(t \mid \zeta) = \frac{dg_q(t \mid \zeta)}{dt}$.

From (1.6), it is easy to show that

$$\sum_{n=0}^{\infty}\left(\zeta q E^q_{n,\zeta}(x+1) + E^q_{n,\zeta}(x)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left([2]_q x^n\right)\frac{t^n}{n!}.$$

By comparing the coefficients in the both sides of $\frac{t^n}{n!}$ on the above, we develop the following:

$$\zeta q E^q_{n,\zeta}(x+1) + E^q_{n,\zeta}(x) = [2]_q x^n. \tag{2.7}$$

From Theorem 1, we get the following equation (2.8):

$$g_q(t \mid \zeta)E^q_{n+1,\zeta}(x) = g_q(t \mid \zeta)xE^q_{n,\zeta}(x) - g'_q(t \mid \zeta)E^q_{n,\zeta}(x). \tag{2.8}$$

So from above

$$\big(\zeta q e^t + 1\big) E_{n+1,\zeta}^q(x) = \big(\zeta q e^t + 1\big) x E_{n,\zeta}^q(x) - \zeta q e^t E_{n,\zeta}^q(x).$$

Thus, we can write the following equation:

$$\zeta q E_{n+1,\zeta}^q(x+1) + E_{n+1,\zeta}^q(x) = \zeta q(x+1) E_{n,\zeta}^q(x+1) + x E_{n,\zeta}^q(x) - \zeta q E_{n,\zeta}^q(x+1). \tag{2.9}$$

From (2.7), (2.8) and (2.9), we can state the following theorem.

Theorem 2. *For $n \geqslant 0$, then we have*

$$\zeta q E_{n,\zeta}^q(x+1) + E_{n,\zeta}^q(x) = [2]_q x^n. \tag{2.10}$$

Remark 1. Assume that $S_n(x)$ is Sheffer sequence for $(g(t), f(t))$. Then Sheffer identity is introduced by

$$S_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} P_k(y) S_{n-k}(x) = \sum_{k=0}^{n} \binom{n}{k} P_k(x) S_{n-k}(y), \tag{2.11}$$

where $P_k(y) = S_k(y) g(t)$ is associated to $f(t)$ (for details, see [6,24,28]).

On account of (1.4) and (2.11), then we have

$$\begin{aligned} E_{n,\zeta}^q(x+y) &= \sum_{k=0}^{n} \binom{n}{k} P_k(y) S_{n-k}(x) \\ &= \sum_{k=0}^{n} \binom{n}{k} E_{n-k,\zeta}^q(y) x^k. \end{aligned}$$

So we have

$$E_{n,\zeta}^q(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k,\zeta}^q(y) x^k.$$

By (1.4), we easily see for $\alpha (\neq 0) \in \mathbb{C}$:

$$E_{n,\zeta}^q(\alpha x) = \frac{g_q(t \mid \zeta)}{g_q(\frac{t}{\alpha} \mid \zeta)} E_{n,\zeta}^q(x). \tag{2.12}$$

From (1.11) and (2.12), we readily derive for $d \equiv 1 \pmod 2$:

$$\frac{g_q(t \mid \zeta)}{g_q(\frac{t}{d} \mid \zeta)} E_{n,\zeta}^q(x) = \frac{d^n}{[d]_{-q}} \sum_{j=0}^{d-1} (-1)^j \zeta^j q^j E_{n,\zeta^d}^{q^d}\left(x + \frac{j}{d}\right).$$

Let us consider the linear functional $f(t)$ that satisfies

$$\langle f(t) \mid p(x)\rangle = \int_{\mathbb{Z}_p} \zeta^{\xi} p(\xi)\, d\mu_{-q}(\xi), \tag{2.13}$$

for all polynomials $p(x)$. From (2.13), we readily see that

$$f(t) = \sum_{n=0}^{\infty} \frac{\langle f(t) \mid x^n\rangle}{n!} t^n = \sum_{n=1}^{\infty} \left(\int_{\mathbb{Z}_p} \zeta^{\xi} \xi^n \, d\mu_{-q}(\xi) \right) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \zeta^{\xi} e^{\xi t}\, d\mu_{-q}(\xi). \tag{2.14}$$

Thus, we have

$$f(t) = \int_{\mathbb{Z}_p} \zeta^{\xi} e^{\xi t}\, d\mu_{-q}(\xi) = \frac{[2]_q}{\zeta q e^t + 1}. \tag{2.15}$$

Therefore, by (2.13) and (2.15), we arrive at the following theorem.

Theorem 3. *For $n \geqslant 0$, then we have*

$$\langle f(t) \mid p(x)\rangle = \int_{\mathbb{Z}_p} \zeta^{\xi} p(\xi)\, d\mu_{-q}(\xi). \tag{2.16}$$

Also,

$$\left\langle \frac{[2]_q}{\zeta q e^t + 1} \,\middle|\, p(x) \right\rangle = \int_{\mathbb{Z}_p} \zeta^{\xi} p(\xi)\, d\mu_{-q}(\xi). \tag{2.17}$$

Obviously that

$$E_{n,\zeta}^q = \left\langle \int_{\mathbb{Z}_p} \zeta^{\xi} e^{\xi t}\, d\mu_{-q}(\xi) \,\middle|\, x^n \right\rangle. \tag{2.18}$$

From (1.6) and (2.18), we see that

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \zeta^{\xi} (x+\xi)^n \, d\mu_{-q}(\xi) \right) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \zeta^{\xi} e^{(x+\xi)t}\, d\mu_{-q}(\xi) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \zeta^{\xi} e^{\xi t}\, d\mu_{-q}(\xi) x^n \right) \frac{t^n}{n!}. \tag{2.19}$$

By (1.6) and (2.20), we see that for $n \in \mathbb{N}^*$:

$$E_{n,\zeta}^q(x) = \int_{\mathbb{Z}_p} \zeta^{\xi} (x+\xi)^n \, d\mu_{-q}(\xi) = \int_{\mathbb{Z}_p} \zeta^{\xi} e^{\xi t}\, d\mu_{-q}(\xi)\, x^n. \tag{2.20}$$

Consequently, we obtain the following theorem.

Theorem 4. *For $p(x) \in \mathcal{P}$, then we have*

$$\int_{\mathbb{Z}_p} \zeta^{\xi} p(x+\xi)\, d\mu_{-q}(\xi) = \int_{\mathbb{Z}_p} \zeta^{\xi} e^{\xi t}\, d\mu_{-q}(\xi)\, p(x) = \frac{[2]_q}{\zeta q e^t + 1} p(x). \tag{2.21}$$

That is,

$$E^q_{n,\zeta}(x) = \int_{\mathbb{Z}_p} \zeta^{\xi} e^{\xi t}\, d\mu_{-q}(\xi)\, x^n = \frac{[2]_q}{\zeta q e^t + 1} x^n. \tag{2.22}$$

For $|1-\zeta|_p < 1$, we introduce weighted q-Euler polynomials of order k as follows:

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} \zeta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k+x)t}\, d\mu_{-q}(\xi_1)\ldots d\mu_{-q}(\xi_k) = \left(\frac{[2]_q}{q\zeta e^t+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E^{(k)}_{n,\zeta}(x \mid q)\frac{t^n}{n!}, \tag{2.23}$$

where, for $x = 0$, $E^{(k)}_{n,\zeta}(0 \mid q) := E^{(k)}_{n,\zeta}(q)$ are called weighted q-Euler numbers of order k.

By (2.23), we have

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k\text{-times}} \zeta^{\xi_1+\cdots+\xi_k} (\xi_1+\cdots+\xi_k+x)^n\, d\mu_{-q}(\xi_1)\ldots d\mu_{-q}(\xi_k) = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\ldots,i_m} \int_{\mathbb{Z}_p} \zeta^{\xi_1}\xi_1^{i_1}\, d\mu_{-q}(\xi_1)\ldots \int_{\mathbb{Z}_p} \zeta^{\xi_k}\xi_k^{i_k}\, d\mu_{-q}(\xi_k) = \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1,\ldots,i_m} E^q_{i_1,\zeta}\cdots E^q_{i_k,\zeta} = E^{(k)}_{n,\zeta}(x \mid q). \tag{2.24}$$

Thanks to (2.23) and (2.24), we have

$$E^{(k)}_{n,\zeta}(x \mid q) = \sum_{l=0}^{n} \binom{n}{l} x^l E^{(k)}_{n-l,\zeta}(q). \tag{2.25}$$

From (2.24) and (2.25), we notice that $E^{(k)}_{n,\zeta}(x \mid q)$ is a monic polynomial of degree n with coefficients in $\mathbb{Q}$. For $k \in \mathbb{N}$, let us assume that

$$g_q^{(k)}(t \mid \zeta) = \Bigg(\underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} \zeta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k)t} \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k) \Bigg)^{-1}$$
$$= \left(\frac{\zeta q e^t + 1}{[2]_q} \right)^k. \tag{2.26}$$

From (2.26), we note that $g_q^{(k)}(t \mid \zeta)$ is an invertible series. On account of (2.23) and (2.26), we readily derive that

$$\frac{1}{g_q^{(k)}(t \mid \zeta)} e^{xt} = \underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} \zeta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k+x)t} \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k)$$
$$= \sum_{n=0}^{\infty} E_{n,\zeta}^{(k)}(x \mid q) \frac{t^n}{n!}. \tag{2.27}$$

Also, we note that

$$t E_{n,\zeta}^{(k)}(x \mid q) = n E_{n-1,\zeta}^{(k)}(x \mid q). \tag{2.28}$$

By (2.27) and (2.28), we easily see that $E_{n,\zeta}^{(k)}(x \mid q)$ is an Appell sequence for $g_q^{(k)}(t \mid \zeta)$. Then, by (2.27) and (2.28), we get the following theorem.

Theorem 5. *For $p(x) \in \mathcal{P}$ and $k \in \mathbb{N}$, we have*

$$\underbrace{\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p}}_{k\text{-times}} \zeta^{\xi_1+\cdots+\xi_k} p(\xi_1 + \cdots + \xi_k + x) \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k) = \left(\frac{[2]_q}{\zeta q e^t + 1} \right)^k p(x). \tag{2.29}$$

In the special case, the weighted q-Euler polynomials of degree k are derived by

$$E_{n,\zeta}^{(k)}(x \mid q) = \left(\frac{2}{\zeta q e^t + 1} \right)^k x^n = \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \zeta^{\xi_1+\cdots+\xi_k} e^{(\xi_1+\cdots+\xi_k)t} \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k) x^n.$$

Thus, we get

$$E_{n,\zeta}^{(k)}(x \mid q) \sim \left(\left(\frac{\zeta q e^t + 1}{[2]_q} \right)^k, t \right).$$

Let us take the linear functional $f^{(k)}(t)$ that satisfies

$$\langle f^{(k)}(t) \mid p(x) \rangle = \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \zeta^{\xi_1+\cdots+\xi_k} p(\xi_1 + \cdots + \xi_k) \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k), \tag{2.30}$$

for all polynomials $p(x)$. Therefore, we compute as follows:

$$\begin{aligned}
f^{(k)}(t) &= \sum_{n=0}^{\infty} \frac{\langle f^{(k)}(t) \mid x^n \rangle}{n!} t^n \\
&= \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \zeta^{\xi_1+\dots+\xi_k} (\xi_1 + \dots + \xi_k)^n \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k) \right) \frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \zeta^{\xi_1+\dots+\xi_k} e^{(\xi_1+\dots+\xi_k)t} \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k) \\
&= \left(\frac{[2]_q}{\zeta q e^t + 1} \right)^k .
\end{aligned}$$

Therefore, the following theorem can be expressed.

Theorem 6. *For $p(x) \in \mathcal{P}$, we have*

$$\begin{aligned}
&\left\langle \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \zeta^{\xi_1+\dots+\xi_k} e^{(\xi_1+\dots+\xi_k)t} \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k) \;\middle|\; p(x) \right\rangle \\
&\quad = \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \zeta^{\xi_1+\dots+\xi_k} p(\xi_1 + \dots + \xi_k) \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k).
\end{aligned}$$

Furthermore,

$$\left\langle \left(\frac{[2]_q}{\zeta q e^t + 1} \right)^k \;\middle|\; p(x) \right\rangle = \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \zeta^{\xi_1+\dots+\xi_k} p(\xi_1 + \dots + \xi_k) \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k).$$

That is,

$$E_{n,\zeta}^{(k)}(q) = \left\langle \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \zeta^{\xi_1+\dots+\xi_k} e^{(\xi_1+\dots+\xi_k)t} \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k) \;\middle|\; x^n \right\rangle.$$

From (1.21), we notice that

$$\begin{aligned}
&\left\langle \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \zeta^{\xi_1+\dots+\xi_k} e^{(\xi_1+\dots+\xi_k)t} \, d\mu_{-q}(\xi_1) \dots d\mu_{-q}(\xi_k) \;\middle|\; x^n \right\rangle \\
&= \sum_{i_1+\dots+i_k=n} \binom{n}{i_1, \dots, i_m} \left\langle \int_{\mathbb{Z}_p} \zeta^{\xi_1} e^{\xi_1 t} \, d\mu_{-q}(\xi_1) \;\middle|\; x^{i_1} \right\rangle \cdots \left\langle \int_{\mathbb{Z}_p} \zeta^{\xi_k} e^{\xi_k t} \, d\mu_{-q}(\xi_k) \;\middle|\; x^{i_k} \right\rangle.
\end{aligned}$$

Therefore, we have

$$E_{n,\zeta}^{(k)}(q) = \sum_{i_1+\dots+i_k=n} \binom{n}{i_1, \dots, i_m} E_{i_1,\zeta}^{q} \cdots E_{i_k,\zeta}^{q}.$$

Remark 2. Our applications for weighted Euler polynomials, weighted q-Euler numbers and weighted q-Euler polynomials of order k seem to be interesting for evaluating at $q=\zeta=1$ which lead to Euler polynomials and Euler polynomials of order k, are defined respectively by

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{xt},$$
$$\sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}=\left(\frac{2}{e^t+1}\right)^k e^{xt}.$$

Also, it is well known that they have representations in terms of fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$E_n(x)=\int_{\mathbb{Z}_p}(x+\xi)^n\,d\mu_{-1}(\xi),$$
$$E_n^{(k)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(\xi_1+\cdots+\xi_k+x)^n\,d\mu_{-1}(\xi_1)\ldots d\mu_{-1}(\xi_k).$$

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