On a Sturm-Liouville Type Problem with Retarded Argument

Erdoğan Şen*[1], Azad Bayramov*[2]

*[1]Department of Mathematics, Faculty of Arts and Science, Namık Kemal University, Tekirdağ, Turkey..

e-mail: abayramov@nku.edu.tr[1],esen@nku.edu.tr[2]

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ABSTRACT. In this work a Sturm-Liouville type problem with retarded argument which contains spectral parameter in the boundary conditions and with transmission conditions at the point of discontinuity are investigated. We obtained asymptotic formulas for the eigenvalues and eigenfunctions.

1 Introduction

We consider the boundary value problem for the differential equation

$$p(x)y''(x)+q(x)y(x-\Delta(x))+\lambda y(x)=0 \tag{1}$$

on $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, with boundary conditions

$$\sqrt{\lambda}y(0)+p_1y'(0)=0, \tag{2}$$

$$d\lambda y(\pi)+y'(\pi)=0, \tag{3}$$

and transmission conditions

$$\gamma_1 y(\frac{\pi}{2}-0)=\delta_1 y(\frac{\pi}{2}+0), \tag{4}$$

$$\gamma_2 y'(\frac{\pi}{2}-0)=\delta_2 y'(\frac{\pi}{2}+0), \tag{5}$$

where $p(x)=p_1^2$ if $x\in\left[0,\frac{\pi}{2}\right)$ and $p(x)=p_2^2$ if $x\in\left(\frac{\pi}{2},\pi\right]$, the real-valued function $q(x)$ is continuous in $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ and has a finite limit $q(\frac{\pi}{2}\pm 0)=\lim_{x\to\frac{\pi}{2}\pm 0}q(x)$, the real valued function $\Delta(x)\geq 0$ continuous in $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ and has a finite limit $\Delta(\frac{\pi}{2}\pm 0)=\lim_{x\to\frac{\pi}{2}\pm 0}\Delta(x)$, $x-\Delta(x)\geq 0$, if $x\in\left[0,\frac{\pi}{2}\right)$; $x-\Delta(x)\geq\frac{\pi}{2}$, if $x\in\left(\frac{\pi}{2},\pi\right]$; λ is a real spectral parameter; $p_1,p_2,\gamma_1,\gamma_2,\delta_1,\delta_2,d$

are arbitrary real numbers and $|\gamma_i| + |\delta_i| \neq 0$ for $i = 1, 2$. Also $\gamma_1\delta_2 p_1 = \gamma_2\delta_1 p_2$ holds.

The present article is devoted to studying the properties of the eigenvalues and eigenfunctions of the boundary value problem (1)-(5). Boundary value problems for ordinary differential equations with a parameter in the boundary conditions in various statements were studied in many articles $[1-10]$.

The article [11] is devoted to the study of the asymptotics of the solutions to the Sturm-Liouville problem with the potential and the spectral parameter having discontinuity of the first kind in the domain of definition of the solution.

Boundary value problems for differential equations of the second order with retarded argument were studied in [12, 13].

The asymptotic formulas for the eigenvalues and eigenfunctions of boundary problem of discontinuous Sturm-Liouville problem with transmission conditions and with the boundary conditions which include spectral parameter were obtained in [14, 15].

In the present considered problem's both boundary conditions involves spectral parameter. The main result of the present paper is Theorem 4 and Theorem 5 on asymptotic formulas for eigenvalues and eigenfunctions.

It must be also noted that some problems with transmission conditions which arise in mechanics (thermal condition problem for a thin laminated plate) were studied in [16].

Let $w_1(x, \lambda)$ be a solution of Eq. (1) on $\left[0, \frac{\pi}{2}\right]$, satisfying the initial conditions

$$w_1(0, \lambda) = p_1, w_1'(0, \lambda) = -\sqrt{\lambda} \tag{6}$$

The conditions (6) define a unique solution of Eq. (1) on $\left[0, \frac{\pi}{2}\right]$([13], p. 12).

After defining above solution we shall define the solution $w_2(x, \lambda)$ of Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$ by means of the solution $w_1(x, \lambda)$ by the initial conditions

$$w_2\left(\frac{\pi}{2}, \lambda\right) = \gamma_1\delta_1^{-1}w_1\left(\frac{\pi}{2}, \lambda\right), \quad \omega_2'(\frac{\pi}{2}, \lambda) = \gamma_2\delta_2^{-1}\omega_1'(\frac{\pi}{2}, \lambda). \tag{7}$$

The conditions (7) are defined as a unique solution of Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$.

Consequently, the function $w(x, \lambda)$ is defined on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ by the equality

$$w(x, \lambda) = \begin{cases} \omega_1(x, \lambda), & x \in [0, \frac{\pi}{2}) \\ \omega_2(x, \lambda), & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

is a such solution of the Eq. (1) on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$;which satisfies boundary conditions and transmission conditions.

Lemma 1 Let $w(x, \lambda)$ be a solution of Eq.(1) and $\lambda > 0$. Then the following integral equations hold:

$$w_1(x,\lambda) = \sqrt{2}p_1 \cos\left(\frac{s}{p_1}x + \frac{\pi}{4}\right) - \frac{1}{s}\int\limits_0^x \frac{q(\tau)}{p_1}\sin\frac{s}{p_1}(x-\tau)\, w_1(\tau - \Delta(\tau), \lambda)\, d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right), \tag{8}$$

$$w_2(x,\lambda) = \frac{\gamma_1}{\delta_1} w_1\left(\frac{\pi}{2}, \lambda\right)\cos\frac{s}{p_2}\left(x - \frac{\pi}{2}\right) + \frac{\gamma_2 p_2 w_1'\left(\frac{\pi}{2}, \lambda\right)}{s\delta_2}\sin\frac{s}{p_2}\left(x - \frac{\pi}{2}\right) - \frac{1}{s}\int\limits_{\pi/2}^x \frac{q(\tau)}{p_2}\sin\frac{s}{p_2}(x-\tau)\, w_2(\tau - \Delta(\tau), \lambda)\, d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right). \tag{9}$$

Proof. To prove this, it is enough to substitute $-\frac{s^2}{p_1^2}\omega_1(\tau,\lambda) - \omega_1''(\tau,\lambda)$ and $-\frac{s^2}{p_2^2}\omega_2(\tau,\lambda) - \omega_2''(\tau,\lambda)$ instead of $-\frac{q(\tau)}{p_1^2}\omega_1(\tau - \Delta(\tau),\lambda)$ and $-\frac{q(\tau)}{p_2^2}\omega_2(\tau - \Delta(\tau),\lambda)$ in the integrals in (8) and (9) respectively and integrate by parts twice. ■

Theorem 1 The problem (1) – (5) can have only simple eigenvalues.

Proof. Let $\widetilde{\lambda}$ be an eigenvalue of the problem (1) – (5) and

$$\widetilde{u}(x,\widetilde{\lambda}) = \begin{cases} \widetilde{u}_1(x,\widetilde{\lambda}), & x \in [0, \frac{\pi}{2}), \\ \widetilde{u}_2(x,\widetilde{\lambda}), & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

be a corresponding eigenfunction. Then from (2) and (6) it follows that the determinant

$$W\left[\widetilde{u}_1(0,\widetilde{\lambda}), w_1(0,\widetilde{\lambda})\right] = \begin{vmatrix} \widetilde{u}_1(0,\widetilde{\lambda}) & p_1 \\ \widetilde{u}_1'(0,\widetilde{\lambda}) & -\sqrt{\lambda} \end{vmatrix} = 0,$$

and by Theorem 2.2.2 in [13] the functions $\widetilde{u}_1(x,\widetilde{\lambda})$ and $w_1(x,\widetilde{\lambda})$ are linearly dependent on $\left[0, \frac{\pi}{2}\right]$. We can also prove that the functions $\widetilde{u}_2(x,\widetilde{\lambda})$ and $w_2(x,\widetilde{\lambda})$ are linearly dependent on $\left[\frac{\pi}{2}, \pi\right]$. Hence

$$\widetilde{u}_1(x,\widetilde{\lambda}) = K_i w_i(x,\widetilde{\lambda}) \qquad (i = 1, 2) \tag{10}$$

for some $K_1 \neq 0$ and $K_2 \neq 0$. We must show that $K_1 = K_2$. Suppose that $K_1 \neq K_2$. From the equalities (4) and (10), we have

$$\begin{aligned} \gamma_1\widetilde{u}(\frac{\pi}{2} - 0, \widetilde{\lambda}) - \delta_1\widetilde{u}(\frac{\pi}{2} + 0, \widetilde{\lambda}) &= \gamma_1\widetilde{u_1}(\frac{\pi}{2}, \widetilde{\lambda}) - \delta_1\widetilde{u_2}(\frac{\pi}{2}, \widetilde{\lambda}) \\ &= \gamma_1 K_1 w_1(\frac{\pi}{2}, \widetilde{\lambda}) - \delta_1 K_2 w_2(\frac{\pi}{2}, \widetilde{\lambda}) \\ &= \gamma_1 K_1 \delta_1 \gamma_1^{-1} w_2(\frac{\pi}{2}, \widetilde{\lambda}) - \delta_1 K_2 w_2(\frac{\pi}{2}, \widetilde{\lambda}) \\ &= \delta_1 (K_1 - K_2)\, w_2(\frac{\pi}{2}, \widetilde{\lambda}) = 0. \end{aligned}$$

Since $\delta_1 (K_1 - K_2) \neq 0$ it follows that

$$w_2 \left(\frac{\pi}{2}, \widetilde{\lambda} \right) = 0. \tag{11}$$

By the same procedure from equality (5) we can derive that

$$w_2^{'} \left(\frac{\pi}{2}, \widetilde{\lambda} \right) = 0. \tag{12}$$

From the fact that $w_2(x, \widetilde{\lambda})$ is a solution of the differential Eq. (1) on $\left[\frac{\pi}{2}, \pi\right]$ and satisfies the initial conditions (11) and (12) it follows that $w_1(x, \widetilde{\lambda}) = 0$ identically on $\left[\frac{\pi}{2}, \pi\right]$ (cf. [13, p. 12, Theorem 1.2.1]).

By using we may also find

$$w_1 \left(\frac{\pi}{2}, \widetilde{\lambda} \right) = w_1^{'} \left(\frac{\pi}{2}, \widetilde{\lambda} \right) = 0.$$

From the latter discussions of $w_2(x, \widetilde{\lambda})$ it follows that $w_1(x, \widetilde{\lambda}) = 0$ identically on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. But this contradicts (6), thus completing the proof. ■

2 An existance theorem

The function $\omega(x,\ \lambda)$ defined in section 1 is a nontrivial solution of Eq. (1) satisfying conditions (2), (4) and (5). Putting $\omega(x,\ \lambda)$ into (3), we get the characteristic equation

$$F(\lambda) \equiv \omega'(\pi,\ \lambda) + d\lambda\omega(\pi,\ \lambda) = 0. \tag{13}$$

By Theorem 1 the set of eigenvalues of boundary-value problem (1)-(5) coincides with the set of real roots of Eq. (13). Let $q_1 = \frac{1}{p_1} \int\limits_0^{\pi/2} |q(\tau)| d\tau$ and $q_2 = \frac{1}{p_2} \int\limits_{\pi/2}^{\pi} q(\tau) d\tau$

Lemma 2 (1) Let $\lambda \geq 4q_1^2$. Then for the solution $w_1(x, \lambda)$ of Eq. (8), the following inequality holds:

$$|w_1(x, \lambda)| \leq 2\sqrt{2}\, |p_1|\,, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{14}$$

(2) Let $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$. Then for the solution $w_2(x, \lambda)$ of Eq. (9), the following inequality holds:

$$|w_2(x, \lambda)| \leq 4\sqrt{2}\, |p_1| \left\{ \left| \frac{\gamma_1}{\delta_1} \right| + \left| \frac{p_2\gamma_2}{4p_1\delta_2} \right| \right\}, \quad x \in \left[\frac{\pi}{2}, \pi\right] \tag{15}$$

Proof. Let $B_{1\lambda} = \max_{\left[0, \frac{\pi}{2}\right]} |w_1(x, \lambda)|$. Then from (8), it follows that, for every $\lambda > 0$, the following inequality holds:

$$B_{1\lambda} \leq \sqrt{2} p_1 + \frac{1}{s} B_{1\lambda} q_1.$$

If $s \geq 2q_1$ we get (14). Differentiating (8) with respect to x, we have

$$w_1'(x,\lambda) = -s\sqrt{2}\sin\left(\frac{sx}{p_1}+\frac{\pi}{4}\right) - \frac{1}{p_1^2}\int\limits_0^x q(\tau)\cos\frac{s}{p_1}(x-\tau)\, w_1(\tau-\Delta(\tau),\lambda)d\tau \tag{16}$$

From (16) and (14), it follows that, for $s \geq 2q_1$, the following inequality holds:

$$|w_1'(x,\lambda)| \leq s\sqrt{2} + 2\sqrt{2}\,|q_1|\,.$$

Hence

$$\frac{|w_1'(x,\lambda)|}{s} \leq \sqrt{2} \tag{17}$$

Let $B_{2\lambda} = \max_{\left[\frac{\pi}{2},\pi\right]}|w_2(x,\lambda)|$. Then from (9), (14) and (17) it follows that, for $s \geq 2q_1$, the following inequalities holds:

$$B_{2\lambda} \leq 4\sqrt{2}\left|\frac{p_1\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{\delta_2}\right|\frac{1}{|p_1q_1|}\sqrt{2} + \frac{B_{2\lambda}}{2},$$
$$B_{2\lambda} \leq 4\sqrt{2}\,|p_1|\left\{\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{4p_1\delta_2}\right|\right\}.$$

Hence if $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$ we get (15). ■

Theorem 2 The problem (1)−(5) has an infinite set of positive eigenvalues.
Proof. Differentiating (9) with respect to x, we get

$$w_2'(x,\lambda) = -\frac{s\gamma_1}{p_2\delta_1}w_1\left(\frac{\pi}{2},\lambda\right)\sin\frac{s}{p_2}\left(x-\frac{\pi}{2}\right) + \frac{\gamma_2 w_1'\left(\frac{\pi}{2},\lambda\right)}{\delta_2}\cos\frac{s}{p_2}\left(x-\frac{\pi}{2}\right)$$
$$-\frac{1}{p_2^2}\int\limits_{\pi/2}^x q(\tau)\cos\frac{s}{p_2}(x-\tau)\, w_2(\tau-\Delta(\tau),\lambda)d\tau. \tag{18}$$

From (8), (9), (13), (16) and (18), we get

$$-\frac{s\gamma_1}{p_2\delta_1}\left(\sqrt{2}p_1\cos\left(\frac{s\pi}{2p_1}+\frac{\pi}{4}\right) - \frac{1}{sp_1}\int\limits_0^{\frac{\pi}{2}} q(\tau)\sin\frac{s}{p_1}(\frac{\pi}{2}-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$
$$\times\sin\frac{s\pi}{2p_2}$$
$$+\frac{\gamma_2}{\delta_2}\left(-s\sqrt{2}\sin\left(\frac{s\pi}{2p_1}+\frac{\pi}{4}\right) - \frac{1}{p_1^2}\int\limits_0^{\frac{\pi}{2}} q(\tau)\cos\frac{s}{p_1}(\frac{\pi}{2}-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right)$$
$$\times\cos\frac{s\pi}{2p_2} - \frac{1}{p_2^2}\int\limits_{\pi/2}^{\pi} q(\tau)\cos\frac{s}{p_2}(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau$$

$$+\lambda d\left(\frac{\gamma_1}{\delta_1}\left[\sqrt{2}p_1\cos\left(\frac{s\pi}{2p_1}+\frac{\pi}{4}\right)-\frac{1}{sp_1}\int\limits_0^{\frac{\pi}{2}}q(\tau)\sin\frac{s}{p_1}(\frac{\pi}{2}-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right]\right.$$

$$\times\cos\frac{s\pi}{2p_2}$$

$$+\frac{\gamma_2 p_2}{\delta_2 s}\left[-s\sqrt{2}\sin\left(\frac{s\pi}{2p_1}+\frac{\pi}{4}\right)-\frac{1}{p_1^2}\int\limits_0^{\frac{\pi}{2}}q(\tau)\cos\frac{s}{p_1}(\frac{\pi}{2}-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right]$$

$$\left.\times\sin\frac{s\pi}{2p_2}-\frac{1}{sp_2}\int\limits_{\frac{\pi}{2}}^{\pi}q(\tau)\sin\frac{s}{p_2}(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau\right)=0 \qquad (19)$$

Let λ be sufficiently large. Then, by (14) and (15), the Eq. (19) may be rewritten in the form

$$s\cos\left(s\pi\frac{p_1+p_2}{2p_1p_2}+\frac{\pi}{4}\right)+O(1)=0 \qquad (20)$$

Obviously, for large s Eq. (20) has an infinite set of roots. Thus the theorem is proved. ■

3 Asymptotic Formulas for Eigenvalues and Eigenfunctions

Now we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following we shall assume that s is sufficiently large. From (8) and (14), we get

$$\omega_1(x,\ \lambda)=O(1) \quad \text{on} \quad [0,\ \frac{\pi}{2}] \qquad (21)$$

From (9) and (15), we get

$$\omega_2(x,\ \lambda)=O(1) \quad \text{on} \quad [\frac{\pi}{2},\ \pi] \qquad (22)$$

The existence and continuity of the derivatives $\omega'_{1s}(x,\lambda)$ for $0\le x\le\frac{\pi}{2}$, $|\lambda|<\infty$, and $\omega'_{2s}(x,\ \lambda)$ for $\frac{\pi}{2}\le x\le\pi$, $|\lambda|<\infty$, follows from Theorem 1.4.1 in [13].

$$\omega'_{1s}(x,\ \lambda)=O(1),\quad x\in[0,\ \frac{\pi}{2}] \quad \text{and} \quad \omega'_{2s}(x,\ \lambda)=O(1),\quad x\in[\ \frac{\pi}{2},\ \pi]. \qquad (23)$$

Theorem 3 Let n be a natural number. For each sufficiently large n, in case 1, there is exactly one eigenvalue of the problem $(1)-(5)$ near $\frac{p_1^2p_2^2}{4(p_1+p_2)^2}(4n-3)^2$
Proof. We consider the expression which is denoted by $O(1)$ in the Eq. (20). If formulas $(21)-(23)$ are taken into consideration, it can be shown by differentiation with respect to s that for large s this expression has bounded derivative.It

is obvious that for large s the roots of Eq. (20) are situated close to entire numbers. We shall show that, for large n, only one root (20) lies near to each $\frac{p_1^2 p_2^2}{4(p_1+p_2)^2}(4n-3)^2$. We consider the function $\phi(s) = s\cos\left(s\pi\frac{p_1+p_2}{2p_1p_2}+\frac{\pi}{4}\right)+O(1)$. Its derivative, which has the form

$$\phi'(s) = \cos\left(s\pi\frac{p_1+p_2}{2p_1p_2}+\frac{\pi}{4}\right) - s\pi\frac{p_1+p_2}{2p_1p_2}\sin\left(s\pi\frac{p_1+p_2}{2p_1p_2}+\frac{\pi}{4}\right)+O(1),$$

does not vanish for s close to n for sufficiently large n. Thus our assertion follows by Rolle's Theorem. ■

Let n be sufficiently large. In what follows we shall denote by $\lambda_n = s_n^2$ the eigenvalue of the problem (1) – (5) situated near $\frac{p_1^2 p_2^2}{4(p_1+p_2)^2}(4n-3)^2$. We set $s_n = \frac{p_1p_2(4n-3)}{2(p_1+p_2)} + \delta_n$. From (20) it follows that $\delta_n = O\left(\frac{1}{n}\right)$. Consequently

$$s_n = \frac{p_1p_2\,(4n-3)}{2\,(p_1+p_2)} + O\big(\frac{1}{n}\big). \tag{24}$$

The formula (24) make it possible to obtain asymptotic expressions for eigenfunction of the problem (1) – (5). From (8), (16) and (21), we get

$$\omega_1(x,\ \lambda) = \sqrt{2}p_1\cos\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)+O\big(\frac{1}{s}\big), \tag{25}$$

$$\omega_1^{'}(x,\ \lambda) = -\sqrt{2}s\sin\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)+O(1). \tag{26}$$

From (9), (22), (25) and (26), we get

$$\omega_2(x,\ \lambda) = \frac{\sqrt{2}\gamma_1p_1}{\delta_1}\cos\left(\frac{s\pi(p_2-p_1)}{2p_1p_2}+\frac{sx}{p_2}+\frac{\pi}{4}\right)+O\big(\frac{1}{s}\big) \tag{27}$$

By putting (24) in the (25) and (27), we derive that

$$u_{1n} = w_1\,(x,\lambda_n) = \sqrt{2}p_1\cos\left(\frac{p_2\,(4n-3)\,x}{2\,(p_1+p_2)}+\frac{\pi}{4}\right)+O\big(\frac{1}{n}\big),$$

$$u_{2n} = w_2\,(x,\lambda_n) = \frac{\sqrt{2}p_1\gamma_1}{\delta_1}\cos\left(\frac{p_1\,(4n-3)\,x}{2\,(p_1+p_2)}+\frac{\pi}{4}\left(1+\frac{(p_2-p_1)\,(4n-3)}{4\,(p_1+p_2)}\right)\right)+O\big(\frac{1}{n}\big).$$

Hence the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x) = \begin{cases} \sqrt{2}p_1\cos\left(\frac{p_2(4n-3)x}{2(p_1+p_2)}+\frac{\pi}{4}\right)+O\big(\frac{1}{n}\big) & \text{for} \quad x\in[0,\frac{\pi}{2}), \\ \frac{\sqrt{2}p_1\gamma_1}{\delta_1}\cos\left(\frac{p_1(4n-3)x}{2(p_1+p_2)}+\frac{\pi}{4}\left(1+\frac{(p_2-p_1)(4n-3)}{4(p_1+p_2)}\right)\right)+O\big(\frac{1}{n}\big) & \text{for} \quad x\in(\frac{\pi}{2},\pi]. \end{cases}$$

Under some additional conditions the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

a) The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$ and have finite limits $q'(\frac{\pi}{2}\pm 0)=\lim\limits_{x\to\frac{\pi}{2}\pm 0}q'(x)$ and $\Delta''(\frac{\pi}{2}\pm 0)=\lim\limits_{x\to\frac{\pi}{2}\pm 0}\Delta''(x)$, respectively.
b) $\Delta'(x)\leq 1$ in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$, $\Delta(0)=0$ and $\lim\limits_{x\to\frac{\pi}{2}+0}\Delta(x)=0$.

By using b), we have

$$x-\Delta(x)\geq 0,\quad \text{for } x\in[0,\frac{\pi}{2})\quad\text{and}\quad x-\Delta(x)\geq\frac{\pi}{2},\quad \text{for } x\in(\frac{\pi}{2},\pi] \tag{28}$$

From (25), (27) and (28) we have

$$w_1\left(\tau-\Delta\left(\tau\right),\lambda\right)=\sqrt{2}p_1\cos\left(\frac{s}{p_1}\left(\tau-\Delta\left(\tau\right)\right)+\frac{\pi}{4}\right)+O(\frac{1}{s}), \tag{29}$$

$$w_2\left(\tau-\Delta\left(\tau\right),\lambda\right)=\frac{\sqrt{2}p_1\gamma_1}{\delta_1}\cos\left(\frac{s\pi(p_2-p_1)}{2p_1p_2}+\frac{s\left(\tau-\Delta\left(\tau\right)\right)}{p_2}+\frac{\pi}{4}\right)+O(\frac{1}{s}). \tag{30}$$

Putting these expressions into (19), we have

$$\begin{aligned}
0=&-\frac{\gamma_2}{\delta_2}\sin\left(\frac{s\pi\left(p_1+p_2\right)}{2p_1p_2}+\frac{\pi}{4}\right)+\frac{sd\gamma_1p_1}{\delta_1}\cos\left(\frac{s\pi\left(p_1+p_2\right)}{2p_1p_2}+\frac{\pi}{4}\right)\\
&-\frac{d\gamma_1}{p_1\delta_1}\left[\sin\left(\frac{s\pi\left(p_1+p_2\right)}{2p_1p_2}\right)\int\limits_0^{\pi/2}\frac{\sqrt{2}q\left(\tau\right)}{2}\cos\left(\frac{s\left(2\tau-\Delta(\tau)\right)}{p_1}+\frac{\pi}{4}\right)d\tau\right.\\
&-\cos\left(\frac{s\pi\left(p_1+p_2\right)}{2p_1p_2}\right)\int\limits_0^{\pi/2}\frac{\sqrt{2}q\left(\tau\right)}{2}\sin\left(\frac{s\left(2\tau-\Delta(\tau)\right)}{p_1}+\frac{\pi}{4}\right)d\tau\\
&+\sin\left(\frac{s\pi\left(p_1+p_2\right)}{2p_1p_2}\right)\sin\int\limits_0^{\pi/2}\frac{\sqrt{2}q\left(\tau\right)}{2}\cos\left(\frac{s\Delta(\tau)}{p_1}-\frac{\pi}{4}\right)d\tau\\
&\left.-\cos\left(\frac{s\pi\left(p_1+p_2\right)}{2p_1p_2}\right)\int\limits_0^{\pi/2}\frac{\sqrt{2}q\left(\tau\right)}{2}\sin\left(\frac{s\Delta(\tau)}{p_1}-\frac{\pi}{4}\right)d\tau\right]\\
&-\frac{d\sqrt{2}\gamma_1}{p_2\delta_1}\left[\sin\left(s\pi\frac{3p_1-p_2}{2p_1p_2}\right)\int\limits_{\pi/2}^{\pi}\frac{\sqrt{2}q\left(\tau\right)}{2}\cos\left(\frac{s(2\tau-\Delta(\tau))}{p_2}+\frac{\pi}{4}\right)d\tau\right.
\end{aligned}$$

$$-\cos\left(s\pi\frac{3p_1-p_2}{2p_1p_2}\right)\int\limits_{\pi/2}^{\pi}\frac{\sqrt{2}q(\tau)}{2}\sin\left(\frac{s(2\tau-\Delta(\tau))}{p_2}+\frac{\pi}{4}\right)d\tau$$

$$+\sin\left(\frac{s\pi(p_1+p_2)}{2p_1p_2}\right)\int\limits_{\pi/2}^{\pi}\frac{\sqrt{2}q(\tau)}{2}\cos\left(\frac{s\Delta(\tau)}{p_2}-\frac{\pi}{4}\right)d\tau$$

$$\left.-\cos\left(\frac{s\pi(p_1+p_2)}{2p_1p_2}\right)\int\limits_{\pi/2}^{\pi}\frac{\sqrt{2}q(\tau)}{2}\sin\left(\frac{s\Delta(\tau)}{p_2}-\frac{\pi}{4}\right)d\tau\right]+O\left(\frac{1}{s}\right) \tag{31}$$

Let

$$A(x,\ s,\ \Delta(\tau))=\int\limits_{0}^{x}\frac{\sqrt{2}q(\tau)}{2}\sin\left(\frac{s\Delta(\tau)}{p_1}-\frac{\pi}{4}\right)d\tau,$$

$$B(x,\ s,\ \Delta(\tau))=\int\limits_{0}^{x}\frac{\sqrt{2}q(\tau)}{2}\cos\left(\frac{s\Delta(\tau)}{p_1}-\frac{\pi}{4}\right)d\tau) \tag{32}$$

It is obviously that these functions are bounded for $0\le x\le\frac{\pi}{2}$, $0<s<\infty$. Let

$$C(x,\ s,\ \Delta(\tau))=\int\limits_{\pi/2}^{x}\frac{\sqrt{2}q(\tau)}{2}\sin\left(\frac{s\Delta(\tau)}{p_2}-\frac{\pi}{4}\right)d\tau,$$

$$D(x,\ s,\ \Delta(\tau))=\int\limits_{\pi/2}^{x}\frac{\sqrt{2}q(\tau)}{2}\cos\left(\frac{s\Delta(\tau)}{p_2}-\frac{\pi}{4}\right)d\tau \tag{33}$$

It is obviously that these functions are bounded for $\frac{\pi}{2}\le x\le\pi$, $0<s<\infty$. Under the conditions a) and b) the following formulas

$$\int\limits_{0}^{x}\frac{\sqrt{2}q(\tau)}{2}\cos\left(\frac{s(2\tau-\Delta(\tau))}{p_1}+\frac{\pi}{4}\right)d\tau=O\left(\frac{1}{s}\right),$$

$$\int\limits_{0}^{x}\frac{\sqrt{2}q(\tau)}{2}\sin\left(\frac{s(2\tau-\Delta(\tau))}{p_1}+\frac{\pi}{4}\right)d\tau=O\left(\frac{1}{s}\right),$$

$$\int\limits_{\pi/2}^{x}\frac{\sqrt{2}q(\tau)}{2}\cos\left(\frac{s(2\tau-\Delta(\tau))}{p_2}+\frac{\pi}{4}\right)d\tau=O\left(\frac{1}{s}\right),$$

$$\int\limits_{\pi/2}^{x}\frac{\sqrt{2}q(\tau)}{2}\sin\left(\frac{s(2\tau-\Delta(\tau))}{p_2}+\frac{\pi}{4}\right)d\tau=O\left(\frac{1}{s}\right). \tag{34}$$

can be proved by the same technique in Lemma 3.3.3 in [13]. From (31), (32), (33), (34) we have

$$\cot\left(\frac{s\pi\left(p_1+p_2\right)}{2p_1p_2}+\frac{\pi}{4}\right)=\frac{1}{s}\left[\frac{\gamma_2}{\delta_2}+\frac{d\gamma_1 B\left(x,s,\Delta\left(\tau\right)\right)}{p_1\delta_1}+\frac{d\gamma_1 D\left(x,s,\Delta\left(\tau\right)\right)}{p_2\delta_1}\right]+O\left(\frac{1}{s^2}\right)$$

Now using $s_n=\frac{p_1p_2(4n-3)}{2(p_1+p_2)}+\delta_n$ we get

$$\cot\left(\frac{\left(2n-1\right)\pi}{2}+\frac{\pi\left(p_1+p_2\right)\delta_n}{2p_1p_2}\right)=-\tan\frac{\pi\left(p_1+p_2\right)\delta_n}{2p_1p_2}=\frac{2\left(p_1+p_2\right)}{p_1p_2\left(4n-3\right)}$$

$$\times\left[\frac{\gamma_2}{\delta_2}+\frac{d\gamma_1 B\left(\frac{\pi}{2},\frac{p_1p_2(4n-3)}{2(p_1+p_2)},\Delta\left(\tau\right)\right)}{p_1\delta_1}+\frac{d\gamma_1 D\left(\pi,\frac{p_1p_2(4n-3)}{2(p_1+p_2)},\Delta\left(\tau\right)\right)}{p_2\delta_1}\right]+O\left(\frac{1}{n^2}\right)$$

and finally

$$\delta_n=\frac{4}{\left(4n-3\right)\pi}\left[\frac{\gamma_2}{\delta_2}+\frac{d\gamma_1 B\left(\frac{\pi}{2},\frac{p_1p_2(4n-3)}{2(p_1+p_2)},\Delta\left(\tau\right)\right)}{p_1\delta_1}+\frac{d\gamma_1 D\left(\pi,\frac{p_1p_2(4n-3)}{2(p_1+p_2)},\Delta\left(\tau\right)\right)}{p_2\delta_1}\right]$$
$$+O\left(\frac{1}{n^2}\right)$$

Thus

$$s_n=\frac{p_1p_2\left(4n-3\right)}{2\left(p_1+p_2\right)}+\frac{4}{\left(4n-3\right)\pi}\left[\frac{\gamma_2}{\delta_2}+\frac{d\gamma_1 B\left(\frac{\pi}{2},\frac{p_1p_2(4n-3)}{2(p_1+p_2)},\Delta\left(\tau\right)\right)}{p_1\delta_1}\right.$$
$$\left.+\frac{d\gamma_1 D\left(\pi,\frac{p_1p_2(4n-3)}{2(p_1+p_2)},\Delta\left(\tau\right)\right)}{p_2\delta_1}\right]+O\left(\frac{1}{n^2}\right)\qquad(35)$$

Thus, we have proven the following theorem.

Theorem 4 If conditions a) and b) are satisfied then, the positive eigenvalues $\lambda_n=s_n^2$ of the problem (1)-(5) have the (35) asymptotic representation for $n\to\infty$.

We now may obtain a sharper asymptotic formula for the eigenfunctions. From (8) and (29)

$$w_1(x,\lambda)=\sqrt{2}p_1\cos\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)$$
$$-\frac{\sqrt{2}}{sp_1}\int_0^x q\left(\tau\right)\sin\left(\frac{s}{p_1}\left(x-\tau\right)\right)\cos\left(\frac{s\left(\tau-\Delta\left(\tau\right)\right)}{p_1}+\frac{\pi}{4}\right)d\tau+O\left(\frac{1}{s^2}\right),$$

$$w_1(x,\lambda)=\sqrt{2}p_1\cos\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)-\frac{\sqrt{2}}{2sp_1}\int\limits_0^x q(\tau)\left[\left[\sin\left(\frac{sx}{p_1}-\frac{\pi}{4}\right)\cos\left(\frac{s(2\tau-\Delta(\tau))}{p_1}\right)\right.\right.$$
$$-\cos\left(\frac{sx}{p_1}-\frac{\pi}{4}\right)\sin\left(\frac{s(2\tau-\Delta(\tau))}{p_1}\right)\Bigg]+\Bigg[\sin\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)\cos\left(\frac{s\Delta(\tau)}{p_1}\right)$$
$$\left.\left.-\cos\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)\sin\left(\frac{s\Delta(\tau)}{p_1}\right)\right]\right]+O\left(\frac{1}{s^2}\right).$$

Thus, from $(32),(33)$ and (34)

$$w_1(x,\lambda)=\cos\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)\left[\sqrt{2}p_1+\frac{A(x,\ s,\ \Delta(\tau))}{sp_1}\right]$$
$$-\frac{\sin\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)B(x,\ s,\ \Delta(\tau))}{sp_1}+O\left(\frac{1}{s^2}\right). \qquad (36)$$

Replacing s by s_n and using (35) for $x\in\left[0,\frac{\pi}{2}\right)$ we have

$$u_{1n}(x)=\cos\left(\frac{p_2(4n-3)x}{2(p_1+p_2)}+\frac{\pi}{4}\right)\left[\sqrt{2}p_1+\frac{2(p_1+p_2)A\left(x,\ \frac{p_1p_2(4n-3)}{2(p_1+p_2)},\ \Delta(\tau)\right)}{p_1^2p_2(4n-3)}\right]$$
$$-\sin\left(\frac{p_2(4n-3)x}{2(p_1+p_2)}+\frac{\pi}{4}\right)\left[\frac{4\sqrt{2}}{(4n-3)\pi}\left[\frac{p_1\gamma_2}{\delta_2}+\frac{d\gamma_1B\left(\frac{\pi}{2},\frac{p_1p_2(4n-3)}{2(p_1+p_2)},\Delta(\tau)\right)}{\delta_1}\right.\right.$$
$$\left.\left.+\frac{dp_1\gamma_1D\left(\pi,\frac{p_1p_2(4n-3)}{2(p_1+p_2)},\Delta(\tau)\right)}{p_2\delta_1}\right]\right]+O\left(\frac{1}{n^2}\right). \qquad (37)$$

From $(16),(29)$ and (32), we have

$$\frac{w_1^{'}(x,\lambda)}{s}=-\sqrt{2}\sin\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)-\frac{B(x,\ s,\ \Delta(\tau))\cos\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)}{p_1^2s}$$
$$-\frac{A(x,\ s,\ \Delta(\tau))\sin\left(\frac{sx}{p_1}+\frac{\pi}{4}\right)}{p_1^2s}+O\left(\frac{1}{s^2}\right)\ ,x\in\left(0,\frac{\pi}{2}\right] \qquad (38)$$

From (9), (30), (34), (36) and (38) we have

$$
\begin{aligned}
w_2(x,\lambda) = {} & \frac{\gamma_1}{2\delta_1}\left\{\left[\sqrt{2}p_1 + \frac{A\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{sp_1}\right]\right. \\
& \times \left[\cos\left(s\left(\frac{\pi(p_1+p_2)}{2p_1p_2} - \frac{x}{p_2}\right) + \frac{\pi}{4}\right) + \cos\left(s\left(\frac{\pi(p_2-p_1)}{2p_1p_2} + \frac{x}{p_2}\right) + \frac{\pi}{4}\right)\right] \\
& - \left[\sin\left(s\left(\frac{\pi(p_1+p_2)}{2p_1p_2} - \frac{x}{p_2}\right) + \frac{\pi}{4}\right) + \sin\left(s\left(\frac{\pi(p_2-p_1)}{2p_1p_2} + \frac{x}{p_2}\right) + \frac{\pi}{4}\right)\right] \\
& \left. \times \frac{B\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{sp_1}\right\} - \left\{\frac{\gamma_2 p_2\sqrt{2}}{2\delta_2} + \frac{\gamma_2 p_2}{2\delta_2 p_1^2 s}\right\} \\
& \times \left\{-\cos\left(s\left(\frac{\pi(p_2-p_1)}{2p_1p_2} + \frac{x}{p_2}\right) + \frac{\pi}{4}\right) + \cos\left(s\left(\frac{\pi(p_1+p_2)}{2p_1p_2} - \frac{x}{p_2}\right) + \frac{\pi}{4}\right)\right\} \\
& - \frac{\gamma_2 p_2 B\left(\frac{\pi}{2}, s, \Delta(\tau)\right)}{2\delta_2 p_1^2 s} \\
& \times \left\{\sin\left(s\left(\frac{\pi(p_2-p_1)}{2p_1p_2} + \frac{x}{p_2}\right) + \frac{\pi}{4}\right) - \sin\left(s\left(\frac{\pi(p_1+p_2)}{2p_1p_2} - \frac{x}{p_2}\right) + \frac{\pi}{4}\right)\right\} \\
& - \frac{\gamma_1 p_1}{sp_2\delta_1}\left\{D\left(x, s, \Delta(\tau)\right)\sin s\left(\frac{x}{p_2} + \frac{\pi(p_2-p_1)}{2p_1p_2}\right)\right. \\
& \left. - C\left(x, s, \Delta(\tau)\right)\cos s\left(\frac{x}{p_2} + \frac{\pi(p_2-p_1)}{2p_1p_2}\right)\right\} + O\left(\frac{1}{s^2}\right), \quad x \in \left(\frac{\pi}{2}, \pi\right].
\end{aligned}
$$

Now, replacing s by s_n and using (35), we have

$$
u_{2n}(x) = \frac{\gamma_1}{2\delta_1}\left\{\left[(-1)^{n+1}\sin\left(\frac{(4n-3)p_1x}{2(p_1+p_2)}\right) + \cos\left(\frac{(4n-3)(p_2-p_1)\pi}{4(p_1+p_2)} + \frac{(4n-3)p_1x}{2(p_1+p_2)} + \frac{\pi}{4}\right)\right]\right.
$$

$$\times\left[\sqrt{2}p_1+\frac{2\,(p_1+p_2)\,A\left(\frac{\pi}{2},\frac{(4n-3)p_1p_2}{2(p_1+p_2)},\Delta(\tau)\right)\right)}{(4n-3)p_1^2p_2}\right]+\left[(-1)^n\cos\left(\frac{(4n-3)p_1x}{2\,(p_1+p_2)}\right)-\right. \tag{1}$$

$$\left.\sin\left(\frac{(4n-3)\,(p_2-p_1)\,\pi}{4\,(p_1+p_2)}+\frac{(4n-3)p_1x}{2\,(p_1+p_2)}+\frac{\pi}{4}\right)\right]\left[\frac{2\,(p_1+p_2)\,B\left(\frac{\pi}{2},\frac{(4n-3)p_1p_2}{2(p_1+p_2)},\Delta(\tau)\right)\right)}{(4n-3)p_1^2p_2}\right]\Bigg\}$$

$$-\frac{\sqrt{2}\gamma_2p_2}{2\delta_2}\left\{(-1)^{n+1}\sin\left(\frac{(4n-3)p_1x}{2\,(p_1+p_2)}\right)+\frac{4}{(4n-3)\pi}\left[(-1)^n\cos\left(\frac{(4n-3)p_1x}{2\,(p_1+p_2)}\right)+\right.\right.$$

$$\left.\times\sin\left(\frac{(4n-3)\,(p_2-p_1)\,\pi}{4\,(p_1+p_2)}+\frac{(4n-3)p_1x}{2\,(p_1+p_2)}+\frac{\pi}{4}\right)\right]\left[\frac{\gamma_2}{\delta_2}+\frac{d\gamma_1B\left(\frac{\pi}{2},\frac{(4n-3)p_1p_2}{2(p_1+p_2)},\Delta(\tau)\right)\right)}{p_1\delta_1}+\right.$$

$$\left.\left.\frac{d\gamma_1D\left(\pi,\frac{(4n-3)p_1p_2}{2(p_1+p_2)},\Delta(\tau)\right)\right)}{p_2\delta_1}\right]+\cos\left(\frac{(4n-3)\,(p_2-p_1)\,\pi}{4\,(p_1+p_2)}+\frac{(4n-3)p_1x}{2\,(p_1+p_2)}+\frac{\pi}{4}\right)\right\}$$

$$+\frac{\gamma_2\,(p_1+p_2)}{\delta_2p_1^3(4n-3)}B\left(\frac{\pi}{2},\frac{(4n-3)p_1p_2}{2\,(p_1+p_2)},\Delta(\tau)\right)\right)\left\{(-1)^n\cos\left(\frac{(4n-3)p_1x}{2\,(p_1+p_2)}\right)\right.$$

$$\left.-\sin\left(\frac{(4n-3)\,(p_2-p_1)\,\pi}{4\,(p_1+p_2)}+\frac{(4n-3)p_1x}{2\,(p_1+p_2)}+\frac{\pi}{4}\right)\right\}+\frac{2\gamma_2\,(p_1+p_2)\,A\left(\frac{\pi}{2},\frac{(4n-3)p_1p_2}{2(p_1+p_2)},\Delta(\tau)\right)}{\delta_2p_1^3(4n-3)}$$

$$\times\left\{(-1)^n\sin\left(\frac{(4n-3)p_1x}{2\,(p_1+p_2)}\right)+\cos\left(\frac{(4n-3)\,(p_2-p_1)\,\pi}{4\,(p_1+p_2)}+\frac{(4n-3)p_1x}{2\,(p_1+p_2)}+\frac{\pi}{4}\right)\right\}-$$

$$\frac{2\gamma_1\,(p_1+p_2)}{(4n-3)\delta_1p_2^2}\left\{\sin\left(\frac{(4n-3)\,(p_2-p_1)\,\pi}{4\,(p_1+p_2)}+\frac{(4n-3)p_1x}{2\,(p_1+p_2)}+\frac{\pi}{4}\right)D\left(x,\frac{(4n-3)p_1p_2}{2\,(p_1+p_2)},\Delta(\tau)\right)\right.$$

$$\left.-C\left(x,\frac{(4n-3)p_1p_2}{2\,(p_1+p_2)},\Delta(\tau)\right)\cos\left(\frac{(4n-3)\,(p_2-p_1)\,\pi}{4\,(p_1+p_2)}+\frac{(4n-3)p_1x}{2\,(p_1+p_2)}+\frac{\pi}{4}\right)\right\}+O(\frac{1}{n^2}). \tag{39}$$

Thus, we have proven the following theorem.

Theorem 5 If conditions a) and b) are satisfied then, the eigenfunctions$u_n(x)$of the problem (1)-(5) have the following asymptotic representation for $n\to\infty$:

$$u_n(x)=\begin{cases} u_{1n}(x) & \text{for} \quad x\in[0,\frac{\pi}{2}) \\ \\ u_{2n}(x) & \text{for} \quad x\in(\frac{\pi}{2},\pi] \end{cases}$$

where $u_{1n}(x)$ and $u_{2n}(x)$ defined as in (37) and (39) respectively.

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