

LEIBNIZ-DIRAC STRUCTURES AND NONCONSERVATIVE SYSTEMS WITH CONSTRAINTS

ÜNVER ÇİFTÇİ

Department of Mathematics, Namık Kemal University
59030 Tekirdağ, Turkey

and

Johann Bernoulli Institute for Mathematics and Computer Science
University of Groningen, PO Box 407
Groningen, The Netherlands

(Communicated by Manuel de León)

ABSTRACT. Although conservative Hamiltonian systems with constraints can be formulated in terms of Dirac structures, a more general framework is necessary to cover also dissipative systems such as gradient and metriplectic systems with constraints. We define Leibniz-Dirac structures which lead to a natural generalization of Dirac and Riemannian structures, for instance. From modeling point of view, Leibniz-Dirac structures make it easy to formulate implicit dissipative Hamiltonian systems. We give their exact characterization in terms of vector bundle maps from the tangent bundle to the cotangent bundle and vice versa. Physical systems which can be formulated in terms of Leibniz-Dirac structures are discussed.

1. Introduction. Dirac structures embody a number of geometric structures such as symplectic, Poisson, foliation, complex geometries [11, 15]. Since their first introduction there have been a great number of work done over the years, which is still growing. One of the most striking features of Dirac structures is that they can give a geometric picture of Hamiltonian systems with constraints, holonomic or nonholonomic [16]. Nevertheless, Dirac structures are insufficient in formulating non-conservative Hamiltonian systems such as gradient systems and systems with damping. In that sense, recently some attempts have been done to put these systems into a rather Hamiltonian form.

For example in [3], a generalization of Dirac structures is given in terms of an inner product of split sign on the Pontryagin bundle instead of the natural symmetric pairing. We specify this definition in order to cover the physical examples which are aimed to be put into the Hamiltonian context. In [20], the authors use the notion of Leibniz structures [14] which is a generalization of Poisson structures, whose tensor is not necessarily skew-symmetric. Our approach is quite similar but we work on the Pontryagin bundle and we also deal with systems with constraints on the manifold. In [19], dissipative Hamiltonian systems with constraints are studied with Dirac's original method of reduced brackets. For other recent work on the generalizations of the conservative Hamiltonian systems we refer to [2, 9] and the references therein. Our motivation in this paper is to give a generalization of

2010 *Mathematics Subject Classification.* 53D05, 70F25, 70G45.

Key words and phrases. Dirac manifolds, dissipative Hamiltonian systems, gradient systems.

Dirac structures and study their geometric features in order to construct a general framework of non-conservative Hamiltonian systems.

We define Leibniz-Dirac structures by weakening the defining properties of Dirac structures as follows: Let V be a vector space with its dual denoted by V^* . A subspace $L \subset V \oplus V^*$ is called a Dirac structure if it is maximally isotropic under the symmetric pairing

$$\langle (v_1, \eta_1), (v_2, \eta_2) \rangle_+ = \frac{1}{2}(\langle \eta_1 | v_2 \rangle + \langle \eta_2 | v_1 \rangle)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in V \oplus V^*$, where $\langle | \rangle$ denotes the natural pairing between vectors and co-vectors. As a result, it is shown in [11] for a Dirac structure that the equations, which are called the *characteristic equations*,

$$\rho(L)^\circ = L \cap V^* \quad \text{and} \quad \rho^*(L)^\circ = L \cap V \quad (1)$$

are satisfied, where ρ and ρ^* denote the projections from $V \oplus V^*$ onto the first and second factor respectively, and $^\circ$ stands for the annihilation operator. Accordingly, there exist skew-symmetric linear maps $\Omega : \rho(L) \rightarrow \rho(L)^*$ and $\Pi : \rho^*(L) \rightarrow \rho^*(L)^*$. A Leibniz-Dirac structure (LD structure in short) is defined to be a subspace $L \subset V \oplus V^*$ such that either of the characteristic equations (1) is satisfied. Then it turns out that this definition is equivalent to the existence of a subspace $E \subset V$ (or $F \subset V^*$) and a linear map $\Omega : E \rightarrow E^*$ (or $\Pi : F \rightarrow F^*$).

The definition of LD structures is so broad that many geometric structures can be treated as LD structures. Of course, Dirac structures form a subfamily of LD structures but besides those; metric, metriplectic and Leibniz structures are covered by the LD structures. Having this in mind, developing the basic geometry of LD structures further forms one of the ingredient of the paper, and it can be said that most of the results on LD structures derived in the present work are a refinement of Dirac structures. We see that LD structures share some properties of Dirac structures such as being a Lagrangian subspace for a suitable inner product. Their extensions on manifolds is defined the same way as in the Dirac structures. Several properties of linear and smooth LD structures are discussed.

Another ingredient of the present work is to show that LD structures are general enough to express possibly dissipative implicit Hamiltonian systems with constraints. Dynamics on smooth LD structures is studied in detail and examples are presented to illustrate the theoretical part. Examples show that LD structures are the proper geometric arena for numerous physical systems.

The paper is organized as follows. In Section 2 we define linear LD structures on vector spaces and give their characterization in terms of linear maps in Theorem 2.4. Then it is shown that LD structures are Lagrangian subspaces with respect to suitable symmetric pairings. Smooth LD structures on manifolds are defined in Section 3, where we also relate LD structures to Leibniz structures. In Section 4 we study admissible functions on manifolds with LD structures, then we study Hamiltonian dynamics of LD manifolds. We present several physical examples which can also be given in different formalisms. The paper ends with some conclusions and future questions.

2. Linear Leibniz-Dirac structures. Let V be an n -dimensional vector space and V^* be its dual space. Consider the direct sum space $V \oplus V^*$ and denote the projections from $V \oplus V^*$ onto V and V^* by ρ and ρ^* , respectively. If $L \subset V \oplus V^*$ is a subspace, it is clear that $\ker \rho|_L = L \cap V^*$ and $\ker \rho^*|_L = L \cap V$ (cf. [11]).

Throughout $L \cap V$ (resp. $L \cap V^*$) will be regarded either as a subspace of V (resp. V^*) or $V \oplus V^*$. For a subspace $W \subset V$ we denote the annihilator by W° . We denote by $\langle \eta | v \rangle$ the natural pairing of a co-vector $\eta \in V^*$ and a vector $v \in V$. After the introduction of notational convention we give the following definition.

Definition 2.1. A Leibniz-Dirac structure (LD structure for short) on V is a subspace $L \subset V \oplus V^*$ which satisfies at least one of the following conditions:

$$\rho(L)^\circ = L \cap V^* \quad (2)$$

$$\rho^*(L)^\circ = L \cap V. \quad (3)$$

LD structures satisfying (2) are called forward LD structures, accordingly the ones satisfying (3) are called backward LD structures.

The meaning of the adjectives ‘forward’ and ‘backward’ will be understood after Theorem 2.4. But before we give some immediate conclusions of the definition of LD structures.

Observe that if $L \subset V \oplus V^*$ is a subspace, then we have the following simple results from Linear Algebra:

$$\dim(L \cap V) + \dim(\rho^*(L)) = \dim(L) \quad (4)$$

$$\dim(L \cap V^*) + \dim(\rho(L)) = \dim(L). \quad (5)$$

So, one can conclude

Proposition 1. *Let L be a subspace of $V \oplus V^*$ with $\dim(V) = n$, then the following are satisfied:*

- (i) *If L is a LD structure, then $\dim(L) = n$.*
- (ii) *If L is n -dimensional and $\rho(L)^\circ \subset L \cap V^*$, then L is a forward LD structure on V .*
- (iii) *If L is n -dimensional and $\rho^*(L)^\circ \subset L \cap V$, then L is a backward LD structure on V .*

Now the question arises: When does a subspace $L \subset V \oplus V^*$ satisfies both of characteristic equations? The following proposition gives a partial answer.

First let us recall two bilinear pairings which are of crucial significance in the theory of Dirac structures [11]:

$$\langle (v_1, \eta_1), (v_2, \eta_2) \rangle_{\mp} = \frac{1}{2} (\langle \eta_1 | v_2 \rangle \mp \langle \eta_2 | v_1 \rangle) \quad (6)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in V \oplus V^*$.

Proposition 2. *Let L be a subspace of $V \oplus V^*$. If L is a Lagrangian space with respect to $\langle\langle, \rangle\rangle_+$ or $\langle\langle, \rangle\rangle_-$, then L satisfies both of the characteristic equations.*

Proof. If L is Lagrangian with respect to $\langle\langle, \rangle\rangle_+$ or $\langle\langle, \rangle\rangle_-$ we have

$$\langle \rho^*(L) | \rho(L \cap V) \rangle = \pm \langle \rho^*(L \cap V) | \rho(L) \rangle = 0,$$

and

$$\langle \rho^*(L \cap V^*) | \rho(L) \rangle = \pm \langle \rho^*(L) | \rho(L \cap V^*) \rangle = 0.$$

So, $L \cap V \subset \rho^*(L)^\circ$ and $L \cap V^* \subset \rho(L)^\circ$. A dimension count gives the equalities, since $\dim(L) = n$. \square

Definition 2.2. A LD structure is called a Dirac structure or a symmetric Dirac structure if it is a Lagrangian subspace with respect to $\langle\langle, \rangle\rangle_+$ or $\langle\langle, \rangle\rangle_-$, respectively.

The skew-symmetric inner product \ll, \gg_- is the canonical symplectic form on the symplectic space $V \oplus V^*$, then a symmetric Dirac structure can be simply defined as a Lagrangian subspace of $V \oplus V^*$.

We give a representation of LD structures, which is an extension of a representation of Dirac structures [11].

Theorem 2.3. *Let L be a LD structure on an n -dimensional vector space V , then there exist two linear maps $A : \mathbb{R}^n \rightarrow V$ and $B : \mathbb{R}^n \rightarrow V^*$ such that*

$$\ker A \cap \ker B = \{0\}, \quad (7)$$

and

$$(\operatorname{Im} A)^\circ = B(\ker A) \quad (7a)$$

if L is a backward LD structure and

$$(\operatorname{Im} B)^\circ = A(\ker B) \quad (7b)$$

otherwise.

Conversely, any structure L on V given by

$$L = \{(A(y), B(y)); y \in \mathbb{R}^n\} \subset V \oplus V^* \quad (8)$$

is a LD structure.

Proof. We only prove what is related to forward LD structures and the other case is similar. Let L be a forward LD structure on V . If one chooses a basis for L , then this is equivalent to giving two linear maps $A : \mathbb{R}^n \rightarrow V$ and $B : \mathbb{R}^n \rightarrow V^*$ such that the basis becomes $(A(e_1), B(e_1)), \dots, (A(e_n), B(e_n))$, where e_1, \dots, e_n is the standard basis for \mathbb{R}^n . Since L is n -dimensional, (7) is satisfied. Observe that $L \cap V^* = B(\ker A)$ and $\rho(L) = \operatorname{Im} A$. Then by the defining property (2) of L , the relation (7a) is satisfied.

Conversely, assume that L is given by (8), then by (7) it is n -dimensional, and (7a) implies (2) which concludes the proof. \square

Next we give another representation of LD structures which gives an equivalent picture of the notion of LD structures.

Theorem 2.4. (i) *A forward LD structure on V can be given by a pair (E, Ω) where $E \subset V$ is a subspace and $\Omega : E \rightarrow E^*$ is a linear map.*

(ii) *A backward LD structure on V can be given by a pair (F, Π) where $F \subset V^*$ is a subspace and $\Pi : F \rightarrow F^*$ is a linear map.*

Proof. Only (i) part of the Theorem will be proved, the other part can be proved with a similar reasoning.

For a given pair (E, Ω) define $L \subset V \oplus V^*$ by

$$L = \{(v, \eta); v \in E, \eta - \Omega(v) \in E^\circ\}. \quad (9)$$

It is clear that $\rho(L) = E$ and

$$L \cap V^* = \{\eta; (0, \eta) \in L\} = \{\eta; \eta - \Omega(0) = \eta \in E^\circ\} = E^\circ. \quad (10)$$

Then one concludes Equation 2 which means that L is a forward LD structure.

Conversely, for a given forward LD structure L set $\rho(L) = E$. Then a linear map $\Omega : E \rightarrow E^*$ can be defined for all $x \in L$ by $\Omega(\rho(x)) := \rho^*(x)|_E$. To show that it is well-defined, consider vectors $x = (v, \eta), x' = (v, \eta') \in L$. We need to show that $\eta|_E = \eta'|_E$. It is clear that $(0, \eta - \eta') \in L$ which implies that $\eta - \eta' \in L \cap V^*$. By the condition (2), this is equivalent to saying that $(\eta - \eta')|_E = 0$ or $\eta|_E = \eta'|_E$, as desired. \square

Theorem 2.4 makes clear where the naming ‘forward’ and ‘backward’ LD structures come from.

Next we have a closer look at the structures of the linear maps Ω and Π , so we will be more clear about the motivation of the definition of LD structures. But first note that the kernel of Ω (resp. Π) is $L \cap V$ (resp. $L \cap V^*$).

Let $\Omega^T : E \rightarrow E^*$ be the adjoint map of Ω , i.e.

$$\langle \Omega^T(v_1)|v_2 \rangle := \langle \Omega(v_2)|v_1 \rangle \quad (11)$$

for all $v_1, v_2 \in E$. Then one can define a symmetric linear map $\Omega^+ : E \rightarrow E^*$ and a skew-symmetric linear map $\Omega^- : E \rightarrow E^*$ by

$$\langle \Omega^+(v_1)|v_2 \rangle := \frac{1}{2} (\langle \Omega(v_1)|v_2 \rangle + \langle \Omega^T(v_1)|v_2 \rangle), \quad (12)$$

$$\langle \Omega^-(v_1)|v_2 \rangle := \frac{1}{2} (\langle \Omega(v_1)|v_2 \rangle - \langle \Omega^T(v_1)|v_2 \rangle), \quad (13)$$

respectively. This allows the unique decomposition

$$\Omega = \Omega^+ + \Omega^- \quad (14)$$

which will be of great importance in the sequel. It is also possible to define the unique decomposition of Π into symmetric and skew-symmetric parts:

$$\Pi = \Pi^+ + \Pi^- \quad (15)$$

If L is a LD structure then

$$\langle (v_1, \eta_1), (v_2, \eta_2) \rangle_+ = \langle \Omega^+(v_1)|v_2 \rangle \quad (16)$$

and

$$\langle (v_1, \eta_1), (v_2, \eta_2) \rangle_- = \langle \Omega^-(v_1)|v_2 \rangle \quad (17)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in L$. Therefore the following is concluded.

Corollary 1. *A LD structure L is a Dirac structure (resp. symmetric Dirac structure) if and only if the corresponding linear map Ω given in Theorem 2.4 is purely skew-symmetric (resp. symmetric).*

Remark 1. The converse of the result above is not generally true, that is, (2) and (3) are not sufficient for Ω (or Π) to be symmetric or skew-symmetric. For instance, if Ω is an isomorphism between V and V^* then the characteristic equations are satisfied. Because, in this case $\rho^*(L) = V^*$ and $L \cap V = \{0\}$.

We can further conclude the following result. It was originally given for Dirac structures in [11], and for LD structures the result was used in [3] without proof.

Proposition 3. *Let $L \subset V \oplus V^*$ be a LD structure on V .*

(i) *If L is a forward LD structure then L is maximally isotropic with respect to some inner product \ll, \gg of split sign and of the form*

$$\ll (v_1, \eta_1), (v_2, \eta_2) \gg = \langle \eta_1|v_2 \rangle + \langle \eta_2|v_1 \rangle - 2\Psi(v_1, v_2), \quad (18)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in V \oplus V^*$, where Ψ is a symmetric bilinear form on V .

(ii) *If L is a backward LD structure then L is maximally isotropic with respect to some inner product \ll, \gg of split sign and of the form*

$$\ll (v_1, \eta_1), (v_2, \eta_2) \gg = \langle \eta_1|v_2 \rangle + \langle \eta_2|v_1 \rangle - 2\Phi(\eta_1, \eta_2), \quad (19)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in V \oplus V^*$, where Φ is a symmetric bilinear form on V^* .

Proof. Only (i) is proved as the proof of (ii) is completely analogous. We know by Theorem 2.4 (i) that L corresponds to a pair (E, Ω) where $E \subset V$ is a subspace and $\Omega : E \rightarrow E^*$ is a linear map. Observe that Ω can be extended to whole V which is also denoted by Ω . This gives a symmetric bilinear form Ψ on V defined by

$$\Psi(v_1, v_2) := \frac{1}{2} (\langle \Omega(v_1) | v_2 \rangle + \langle \Omega(v_2) | v_1 \rangle). \quad (20)$$

(We note here that the extension of Ω is not unique so the symmetric bilinear form is not uniquely defined, but this does not change the result.) Then it is straight forward to show that L is isotropic with respect to the symmetric bilinear form in (18). It remains to show that (18) is an inner product of split sign. After choosing a proper basis for $V \oplus V^*$, the result will be clear. Let $\alpha_1, \dots, \alpha_n$ be a basis of V and β_1, \dots, β_n be a basis of V^* such that $\langle \beta_i | \alpha_j \rangle = \delta_i^j$, $i, j = 1, \dots, n$, where δ is the Kronecker symbol. As a basis of $V \oplus V^*$ one can choose $(0, \beta_1), \dots, (0, \beta_n), (\alpha_1, \Omega(\alpha_1)), \dots, (\alpha_n, \Omega(\alpha_n))$, then the matrix associated to the bilinear form in (18) becomes

$$\begin{pmatrix} O_n & I_n \\ I_n & O_n \end{pmatrix},$$

where O_n is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. Accordingly, the basis given by

$$\begin{aligned} y_i &= \frac{\sqrt{2}}{2} [(0, \beta_i) + (\alpha_i, \Omega(\alpha_i))], \\ x_i &= \frac{\sqrt{2}}{2} [(0, \beta_i) - (\alpha_i, \Omega(\alpha_i))] \end{aligned}$$

gives the diagonal form

$$\begin{pmatrix} I_n & O_n \\ O_n & -I_n \end{pmatrix}.$$

Then it is concluded that the bilinear form in (18) has signature (n, n) with $n = \dim(V)$. This concludes the proof. \square

We can deduce from the proof of Proposition 3 that LD structures can be defined as deformations of Dirac structures as follows. Let $\text{Symm}(V)$ and $\text{Symm}(V^*)$ be the additive groups of symmetric bilinear forms on V and V^* respectively, and let $\text{Dir}(V)$, $\text{FLD}(V)$ and $\text{BLD}(V)$ denote the spaces of Dirac structures, forward LD structures and backward LD structures on V respectively. Then we have

Corollary 2. (i) For a given pair $(\psi, L) \in \text{Symm}(V) \times \text{Dir}(V)$ consider the subspace

$$L' = \{(v, \eta + \psi(v)); (v, \eta) \in L\}, \quad (21)$$

then $L' \in \text{FLD}(V)$. Conversely, every $L' \in \text{FLD}(V)$ can be obtained by a pair $(\psi, L) \in \text{Symm}(V) \times \text{Dir}(V)$.

(ii) For a given pair $(\phi, L) \in \text{Symm}(V^*) \times \text{Dir}(V)$ consider the subspace

$$L' = \{(v + \phi(\eta), \eta); (v, \eta) \in L\} \quad (22)$$

then $L' \in \text{BLD}(V)$. Conversely, every $L' \in \text{BLD}(V)$ can be obtained by a pair $(\phi, L) \in \text{Symm}(V^*) \times \text{Dir}(V)$.

Proof. Only (i) will be proved as the proof of (ii) is similar. (i) If $(\psi, L) \in \text{Symm}(V) \times \text{Dir}(V)$, then L' defined by (21) is obviously a LD structure. Conversely, every forward LD structure L has a representation (E, Ω) and Ω can be extended to V . Furthermore Ω can be split into $\Omega = \Omega^+ + \Omega^-$. Then we have

$$L' = \{(v, \eta + \Omega^+(v)); (v, \eta) \in L\}, \quad (23)$$

where L is the Dirac structure given by (E, Ω^-) . \square

The idea behind Corollary 2 is the notion of gauge equivalence of Dirac structures [7]: Let $\text{Bil}(V)$ denote the additive group of skew-symmetric bilinear forms on V then recall that there is an action

$$\tau : \text{Bil}(V) \times \text{Dir}(V) \rightarrow \text{Dir}(V) \quad (24)$$

defined by gauge transformations

$$\tau_B(L) = \{(v, \eta + B(v)); (v, \eta) \in L\}, \quad B \in \text{Bil}(V). \quad (25)$$

Now we address to the question: When a symmetric Dirac structure is also a Dirac structure? But before we recall the definition of a separable Dirac structures [23], a notion which appears as a generalization of Tellegen's theorem in circuit theory. A Dirac structure $L \subset V \oplus V^*$ is a *separable Dirac structure* if

$$\langle \eta_1 | v_2 \rangle = 0, \quad (26)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in L$. It is obtained in [23] that a subspace $L \subset V \oplus V^*$ is a separable Dirac structure if and only if

$$L = K \oplus K^\circ \quad (27)$$

for some subspace $K \subset F$.

We then have

Proposition 4. *A subspace $L \subset V \oplus V^*$ is both a Dirac and a symmetric Dirac structure if and only if it is a separable Dirac structure.*

Proof. L is a Dirac and a symmetric Dirac structure, then

$$\langle \eta_1 | v_2 \rangle + \langle \eta_2 | v_1 \rangle = 0, \quad (28)$$

and

$$\langle \eta_1 | v_2 \rangle - \langle \eta_2 | v_1 \rangle = 0 \quad (29)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in L$, respectively. Then summing these equations gives Equation 26.

Conversely, if L is a separable Dirac structure, it is easily seen by Equation (26) that the symmetric Dirac condition (29) is satisfied trivially. \square

3. Smooth Leibniz-Dirac structures. Let M be a n -dimensional smooth manifold. Consider a smooth subbundle L of the Pontryagin bundle $TM \oplus T^*M$. We denote by the projections from $TM \oplus T^*M$ onto TM and T^*M by ρ and ρ^* , respectively. Definition 2.1 can be given on a manifold as the following.

Definition 3.1. Let L be a smooth vector subbundle of $TM \oplus T^*M$. Then L is called a Leibniz-Dirac structure (LD structure) if either of the following equations holds:

$$\rho(L)^\circ = L \cap T^*M \quad (30)$$

$$\rho^*(L)^\circ = L \cap TM. \quad (31)$$

LD structures satisfying (30) are called forward LD structures, accordingly the ones satisfying (31) are called backward LD structures.

Remark 2. The equations (30) and (31) are not vector bundle equations, in general. That is, the distributions and codistributions included in the equations may not have constant rank. However, (30) implies

$$\rho(L) \subset (L \cap T^*M)^\circ \quad (32)$$

and (31) implies

$$\rho^*(L) \subset (L \cap TM)^\circ \quad (33)$$

with the equality if the relations are vector bundle relations.

By considering the preceding remark we have the following which is similar to the linear case.

Proposition 5. *Let L be a subbundle of $TM \oplus T^*M$ with $\dim(M) = n$, then the following are satisfied:*

- (i) *If L is a LD structure, then $\text{rank}(L) = n$.*
- (ii) *If the rank of L is equal to n and $\rho(L)^\circ \subset L \cap T^*M$ is satisfied as a vector bundle equation, then L is a forward LD structure on M .*
- (iii) *If the rank of L is equal to n and $\rho^*(L)^\circ \subset L \cap TM$ is satisfied as a vector bundle equation, then L is a backward LD structure on M .*

Proof. (i) We only consider forward LD structures, the other case is similar. Since ρ is a vector bundle map, $\rho(L)$ is a smooth distribution. Then there is a set on which $\rho(L)$ has maximal rank. This set is open and dense [6], and on this set $\rho(L)$ and hence $L \cap TM$ are vector bundles. Then the rank of L on these points is n . Since L is a vector bundle one has that $\text{rank}(L(x)) = n$ for all $x \in M$.

(ii) As $\rho(L)$ and $L \cap TM$ have constant rank by the hypothesis, we have the equation

$$\dim(L(x) \cap T_x M) + \dim(\rho^*(L(x))) = \dim(L(x)) \quad (34)$$

for all $x \in M$. Therefore $\rho(L(x))^\circ = L(x) \cap T_x^* M$ for all $x \in M$, then one concludes that $\rho(L)^\circ = L \cap T^*M$.

(iii) As $\rho^*(L)$ and $L \cap T^*M$ have constant rank by the hypothesis, we have the equation

$$\dim(L(x) \cap T_x^* M) + \dim(\rho(L(x))) = \dim(L(x)) \quad (35)$$

for all $x \in M$. Therefore the result follows. \square

We proceed with the relation between LD structures and Lagrangian subbundles of $TM \oplus T^*M$. The two bilinear pairings are defined by

$$\langle (v_1, \eta_1), (v_2, \eta_2) \rangle_{\mp} = \frac{1}{2} (\langle \eta_1 | v_2 \rangle \mp \langle \eta_2 | v_1 \rangle) \quad (36)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in TM \oplus T^*M$.

Proposition 2 extends directly to the following.

Proposition 6. *A subbundle $L \subset TM \oplus T^*M$ satisfies both of the equations (30) and (31) if it is a Lagrangian subbundle with respect to $\langle\langle, \rangle\rangle_+$ or $\langle\langle, \rangle\rangle_-$.*

Definition 3.2. A subbundle $L \subset TM \oplus T^*M$ is called a Dirac structure or a symmetric Dirac structure if it is a Lagrangian subbundle with respect to $\langle\langle, \rangle\rangle_+$ or $\langle\langle, \rangle\rangle_-$.

Now a locally defined representation of LD structures is given as an extension of the linear case given in Theorem 2.3.

Theorem 3.3. *Let L be a LD structure on an n -dimensional manifold M , then there exist two locally defined vector bundle maps $A : M \times \mathbb{R}^n \rightarrow TM$ and $B : M \times \mathbb{R}^n \rightarrow T^*M$ such that for all $m \in M$*

$$\ker A_m \cap \ker B_m = \{0\}, \quad (37)$$

and

$$(Im A_m)^\circ = B_m(\ker A_m) \quad (37a)$$

if L is a backward LD structure and

$$(Im B_m)^\circ = A_m(\ker B_m) \quad (37b)$$

otherwise. Here $A_m : \mathbb{R}^n \rightarrow T_m M$ and $B_m : \mathbb{R}^n \rightarrow T_m^* M$ are the linear maps defined at a fixed $m \in M$.

Conversely, any subbundle L on M given for all $m \in M$ by

$$L(m) = \{(A_m(y), B_m(y)); m \in M, y \in \mathbb{R}^n\} \subset T_m M \oplus T_m^* M \quad (38)$$

is a LD structure.

Proof. Since L is a subbundle, a choice of a local basis of sections for L gives two vector bundle maps $A : M \times \mathbb{R}^n \rightarrow TM$ and $B : M \times \mathbb{R}^n \rightarrow T^*M$. Then the remaining part of the proof is obvious by the proof of Theorem 2.3. \square

Another representation of LD structures is given as the following.

Theorem 3.4. *(i) A forward LD structure L on M such that $\rho(L)$ is a subbundle can be given by a pair (E, Ω) , where $E \subset TM$ is a subbundle and $\Omega : E \rightarrow E^*$ is a vector bundle map.*

(ii) A backward LD structure L on M such that $\rho^(L)$ is a subbundle can be given by a pair (F, Π) , where $F \subset T^*M$ is a subbundle and $\Pi : F \rightarrow F^*$ is a vector bundle map.*

Proof. Only the proof of (i) is given, a similar reasoning holds for the case (ii).

(i) For a given pair (E, Ω) consider $L \subset TM \oplus T^*M$ defined by

$$L = \{(v, \eta); v \in E, \eta - \Omega(v) \in E^\circ\}. \quad (39)$$

Then L is a subbundle since E is a subbundle and Ω is a vector bundle map. It is also easy to see that $\rho(L) = E$ and

$$L \cap T^*M = \{\eta; (0, \eta) \in L\} = \{\eta; \eta - \Omega(0) = \eta \in E^\circ\} = E^\circ. \quad (40)$$

Thus equation (30) is obtained.

Conversely, for a given forward LD structure L set $\rho(L) = E$. Then the map $\Omega : E \rightarrow E^*$ defined for all $x \in L$ by $\Omega(\rho(x)) := \rho^*(x)|_E$ is well-defined by the condition (30). Then it is a vector bundle map, since ρ^* is a vector bundle map. \square

Having an equivalent picture of LD structures in terms of both subbundles of $TM \oplus T^*M$, and pairs (E, Ω) and (F, Π) is very useful as seen in the preceding section. To make use of this equivalent picture we make the following assumption.

Remark 3. In the sequel, any forward (resp. backward) LD structure L will be assumed to be given by a pair (E, Ω) (resp. (F, Π)) in such a way that the characteristic distribution $\rho(L) = E$ (resp. co-distribution $\rho^*(L) = F$) has constant rank.

For a LD structure, as in the linear case, one has the unique decomposition

$$\Omega = \Omega^+ + \Omega^- \quad (41)$$

where Ω^+ is symmetric and Ω^- is skew-symmetric. Similarly one can define the unique decomposition of Π into symmetric and skew-symmetric parts:

$$\Pi = \Pi^+ + \Pi^-. \quad (42)$$

Proposition 7. (i) *A forward LD structure L is locally maximally isotropic with respect to some inner product \ll, \gg of split sign and of the form*

$$\ll (v_1, \eta_1), (v_2, \eta_2) \gg = \langle \eta_1 | v_2 \rangle + \langle \eta_2 | v_1 \rangle - 2\Psi(v_1, v_2), \quad (43)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in TM \oplus T^*M$, where Ψ is a symmetric covariant tensor field on M .

(ii) *A backward LD structure L is maximally isotropic with respect to some inner product \ll, \gg of split sign and of the form*

$$\ll (v_1, \eta_1), (v_2, \eta_2) \gg = \langle \eta_1 | v_2 \rangle + \langle \eta_2 | v_1 \rangle - 2\Phi(\eta_1, \eta_2), \quad (44)$$

for all $(v_1, \eta_1), (v_2, \eta_2) \in TM \oplus T^*M$, where Φ is a symmetric contravariant tensor field on M .

Proof. The point here is that one can extend Ω (resp. Π) to TM (resp. T^*M) locally to define Ψ (resp. Φ). This extension can be done by Tietsze Extension Theorem [1]. The remainder of the proof is a straightforward extension of Proposition 7 when considered pointwise. \square

Dirac structures form a particular subclass of LD structures, which include symplectic, Poisson and foliation geometries. Some other examples of LD structures are discussed below.

Example 1. Let (M, g) be a pseudo-Riemannian manifold. The musical isomorphism $g^\sharp : T^*M \rightarrow TM$ of the pseudo-Riemannian metric g is a vector bundle map. Then the graph of g^\sharp given by

$$L = \{(X, \eta); X = -g^\sharp(\eta)\} \in TM \oplus T^*M \quad (45)$$

defines a LD structure on M which is symmetric. It will be explained in the next section that this setting allows one to study gradient control systems with constraints [3].

Example 2. A vector bundle map $\Pi : T^*M \rightarrow TM$ is called a *Leibniz structure* [20]. Then the graph of Π is a LD structure on M . These structures are shown to model a very large family of physical systems [18, 20]. However, LD structures also allow to add some constraints when modeling physical systems (cf. Section 4).

4. Dynamics on LD manifolds. Dynamic properties of LD structures are given in this section. We first give the notion of admissible functions. The main ingredient of this section is a formulation of dissipative Hamiltonian systems with constraints.

4.1. Admissible functions. Admissible functions on LD manifolds are defined as in the Dirac case [11]. This definition makes sense for only backward LD structures as being a generalization of the Poisson bracket.

Definition 4.1. Let L be a backward LD structure on a manifold M . A function f on M is called an *admissible function* if there exists a vector field X_f such that $(X_f, df) \in L$.

As in the Dirac case [11] we have the following.

Lemma 4.2. *Let L be a backward LD structure on a manifold M . If f and g are admissible functions then fg is also an admissible function.*

Proof. By the hypothesis $(X_f, df), (X_g, dg) \in L$ for some vector fields X_f and X_g on M . Then one computes

$$\begin{aligned} g(X_f, df) + f(X_g, dg) &= (gX_f + fX_g, gdf + fdg) \\ &= (gX_f + fX_g, d(fg)) \in L. \end{aligned}$$

So, fg is an admissible function. \square

Note that if f and g are admissible functions, then fg is an admissible such that $(X_{fg}, d(fg)) \in L$, where $X_{fg} := gX_f + fX_g$.

In accordance with the Dirac case, a bracket $\{\{, \}\}$ on admissible functions on M can be defined by

$$\{\{f, g\}\} = X_f(g) = \langle dg | X_f \rangle \quad (46)$$

for some $(X_f, df), (X_g, dg) \in L$. If (F, Π) is the corresponding backward LD pair to L , then

$$\{\{f, g\}\} = \langle dg | \Pi(df) \rangle. \quad (47)$$

Note that the bracket $\{\{, \}\}$ is well-defined as $\{\{f, g\}\}$ does not depend on X_f and X_g .

The following result is an extension of the Dirac case [11].

Proposition 8. *With the notation above, the bracket $\{\{, \}\}$ on admissible functions satisfy the Leibniz identities:*

$$\{\{fg, h\}\} = f\{\{g, h\}\} + g\{\{f, h\}\} \quad (48)$$

$$\{\{h, fg\}\} = f\{\{h, g\}\} + g\{\{h, f\}\} \quad (49)$$

for all admissible functions f, g, h on M .

Proof. Let $(X_f, df), (X_g, dg), (X_h, dh) \in L$. Then

$$\begin{aligned} \{\{fg, h\}\} &= \langle dh | \Pi(d(fg)) \rangle \\ &= f\langle dh | \Pi(dg) \rangle + g\langle dh | \Pi(df) \rangle \\ &= f\{\{g, h\}\} + g\{\{f, h\}\}, \end{aligned}$$

since Π is a vector bundle map, and

$$\begin{aligned} \{\{h, fg\}\} &= \langle dh | \Pi(d(fg)) \rangle \\ &= f\langle dg | \Pi(dh) \rangle + g\langle df | \Pi(dh) \rangle \\ &= f\{\{h, g\}\} + g\{\{h, f\}\}, \end{aligned}$$

fg is an admissible function. \square

Observe that the bracket $\{\{, \}\}$ splits into a skew-symmetric bracket $\{, \}$ and a symmetric bracket $[,]$. In fact using the splitting (42) one has

$$\begin{aligned} \{\{f, g\}\} &= \langle df | \Pi(dg) \rangle \\ &= \langle df | \Pi^-(dg) \rangle + \langle df | \Pi^+(dg) \rangle \\ &= \{f, g\} + [f, g], \end{aligned}$$

where

$$\{f, g\} := \langle df | \Pi^-(dg) \rangle \quad (50)$$

and

$$[f, g] := \langle df | \Pi^+(dg) \rangle. \quad (51)$$

Remark 4. Note that the bracket of two admissible functions is not again an admissible function, in general. This is so even in the Dirac case, however an integrability condition ensures the closedness of the bracket [11].

A (weak) integrability of LD structures on manifolds is defined in accordance with the one on Dirac structures [11, 12].

Definition 4.3. Let L be a backward LD structure on a manifold M . If $\rho^*(L)^\circ = L \cap TM$ is involutive and the bracket $\{\{, \}\}$ is closed on admissible functions, then L is called a weakly integrable backward LD structure.

Note that in the case of Dirac structures the integrability is equivalent to the above conditions and additionally the Jacobi identity on admissible functions [12]. (We already assume that $\rho^*(L)$ has constant rank by Assumption 3.)

The following shows that the weak integrability definition makes sense.

Proposition 9. *Let L be a weakly integrable backward LD structure on a manifold M . If the foliation of $L \cap TM$ is denoted by Φ and M/Φ is a manifold, then M/Φ inherits a Leibniz structure.*

Proof. Functions on M/Φ can be considered as Φ -invariant functions on M . These functions are the ones $f \in C^\infty(M)$ with $df(T\Phi) = 0$. By the definition these functions correspond to the admissible functions. Therefore by the weak integrability assumption they are closed under the bracket and give rise to an induced bracket on M/Φ , which satisfies the Leibniz identities. \square

4.2. Nonconservative systems with constraints. In this subsection we are concerned with backward LD structures. This is because most of the physical examples we study fall into that category.

Let L be a LD structure on a manifold M , then one can extend the notion of implicit Hamiltonian systems [12] to LD structures as follows.

Definition 4.4. Let $H : M \rightarrow \mathbb{R}$ be a Hamiltonian. The dissipative implicit Hamiltonian system (DIHS) corresponding to (M, L, H) is given by

$$(\dot{x}, dH(x)) \in L(x), \quad x \in M. \quad (52)$$

In this setting, $\rho(L)$ describes the set of admissible flows \dot{x} , that is, the flows of vector fields corresponding to the admissible function dH . The distribution $\rho^*(L)$ describes the set of algebraic constraints. Assume that L is represented by

the pair $(F = \rho^*(L), \Pi)$, then the DIHS corresponding to (M, L, H) has a local representation

$$\begin{aligned}\dot{x} &= \Pi(x) \frac{\partial H}{\partial x}(x) + G(x) \lambda, \\ 0 &= G^T(x) \frac{\partial H}{\partial x}(x),\end{aligned}\tag{53}$$

where $\frac{\partial H}{\partial x}(x)$ stands for the column vector of partial derivatives of H , and $G(x)$ is a full rank matrix with $\text{Im } G(x) = L(x) \cap T_x M$, and λ are Lagrange multipliers corresponding to the algebraic constraints $0 = G^T(x) \frac{\partial H}{\partial x}(x)$ [12].

In terms of brackets one can obtain the equations of motion as

$$\frac{df}{dt} = \dot{x}(f) = \langle df | \dot{x} \rangle = \langle df | \Pi(dH) \rangle = \{\{f, H\}\}\tag{54}$$

for all admissible functions $f \in C^\infty(M)$. Therefore, if the splitting (42) is considered, by (54) one obtains

$$\frac{dH}{dt} = \langle dH | \Pi^+(dH) \rangle = [H, H].\tag{55}$$

If L is a Dirac structure then Equation 55 is nothing but the conservation of energy. For nonconservative systems Equation 55 has several meanings which will be cleared below.

Example 3 (Gradient systems with constraints). Let M be a smooth manifold with a pseudo-Riemannian metric g and let $F \subset T^*M$ be a subbundle. Consider the LD structure

$$L = \{(X, \eta); X + g^\sharp(\eta) \in F^\circ\} \subset TM \oplus T^*M.\tag{56}$$

Let $S : M \rightarrow \mathbb{R}$ be an *entropy function* [18]. Then the gradient system with constraints corresponding to (M, L, S) is defined by

$$(\dot{x}, dS(x)) \in L(x), \quad x \in M.\tag{57}$$

Or, it can be represented by

$$\begin{aligned}\dot{x} &= -g^\sharp(x) \frac{\partial S}{\partial x}(x) + G(x) \lambda, \\ 0 &= G^T(x) \frac{\partial S}{\partial x}(x),\end{aligned}\tag{58}$$

where $G(x)$ is a full rank matrix with $\text{Im } G(x) = L(x) \cap T_x M$ [3].

Then the equations of motions in brackets read

$$\frac{df}{dt} = -\langle df | g^\sharp(dS) \rangle = [f, S]\tag{59}$$

for all admissible functions $f \in C^\infty(M)$. Recall here that the bracket $[,]$ is called the *Beltrami bracket* [10]. Eventually one obtains the equation

$$\frac{dS}{dt} = -\langle dS | g^\sharp(dS) \rangle = [S, S] \leq 0\tag{60}$$

which is called the *entropy equation* [3].

One of the examples of gradient systems with constraints is RCL circuits with excess elements. This topic was studied in [4] by using LD structures, but we believe that it is more convenient to do a study particularly by symmetric Dirac structures.

Example 4 (Metriplectic systems with constraints). Let M be a manifold and $F \subset T^*M$ be a subbundle. Let $P : T^*M \rightarrow TM$ be a Poisson structure and g be (possibly degenerate) Riemannian metric. Set a Leibniz tensor by

$$\Pi = P - g^\sharp \quad (61)$$

then the LD structure given by

$$L = \{(X, \eta); X + \Pi(\eta) \in F^\circ\} \subset TM \oplus T^*M. \quad (62)$$

is called a *metriplectic structure* [18]. If $H : M \rightarrow \mathbb{R}$ is a smooth function, then the system given by

$$(\dot{x}, dH(x)) \in L(x), \quad x \in M. \quad (63)$$

is called a *metriplectic system with constraints*. The corresponding equations of motions take the form

$$\frac{df}{dt} = \langle df | P(dH) \rangle - \langle df | g^\sharp(dH) \rangle = \{f, H\} + [f, H] \quad (64)$$

for all admissible functions $f \in C^\infty(M)$. Then one obtains the equation

$$\frac{dH}{dt} = -\langle dH | g^\sharp(dH) \rangle = [H, H] \leq 0 \quad (65)$$

which describes the dissipation of energy [5, 19].

Now we discuss how to determine the λ in Equation 53, see [12] for details. Assume that the $n \times k$ matrix $G(x)$ has rank $k \leq n$. Then there exists an $(n-k) \times n$ matrix $K(x)$ such that $K(x)G(x) = 0$. Therefore multiplying by $K(x)$ puts Equation 53 to the form

$$\begin{bmatrix} K(x) \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} K(x)\Pi(x) \\ G(x)^T \end{bmatrix} \frac{\partial H}{\partial x}(x). \quad (66)$$

Example 5 (Mechanical systems with damping). Let Q be a manifold (configuration space) and let $q = (q_1, \dots, q_n)$ be a local coordinate system on Q . Consider a Hamiltonian $H(q, p)$ on $M = T^*Q$ where (q, p) is the natural coordinate system on T^*Q . A *mechanical system with damping* [22] subject to k independent kinematic constraints

$$A^T(q)\dot{q} = 0, \quad (67)$$

can be defined by the representation

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} O & I \\ -I & R(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix} + \begin{bmatrix} O \\ A(q) \end{bmatrix} \lambda, \\ O &= \begin{bmatrix} O & A^T(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix}, \end{aligned}$$

where $R(q)$ is a semidefinite matrix. Here the constraint forces $A(q)\lambda$ with $\lambda \in \mathbb{R}^k$ are uniquely determined by the requirement that the constraints (67) have to be satisfied for all time. Since $\text{rank}(A(q)) = k$, one can find an $(n-k) \times n$ matrix $K(q)$ of constant rank $n-k$ such that $K(q)A(q) = 0$. Then the above system assumes the form

$$\begin{bmatrix} I & O \\ O & K(q) \\ O & O \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} O & I \\ -K(q) & K(q)R(q) \\ O & A^T(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix}.$$

In terms of LD structures one can define the system in question as follows. As the matrix $A(q)$ has rank k , its columns span a co-distribution, say G_0 , of constant rank on Q . Set $G := \pi^*(G_0)$ with $\pi : T^*Q \rightarrow Q$ the natural projection. Let $B : T^*(T^*Q) \rightarrow T(T^*Q)$ be the canonical Poisson structure on T^*Q , which has the matrix form

$$\begin{bmatrix} O & I \\ -I & O \end{bmatrix}$$

in the natural coordinates (q, p) . Let $\tilde{R} : T^*(T^*Q) \rightarrow T(T^*Q)$ be the vector bundle map with the matrix

$$\begin{bmatrix} O & O \\ O & -R(q) \end{bmatrix}$$

and set the Leibniz structure given by $\Pi := B - \tilde{R}$. Consider the LD structure

$$L = \{(X, \eta); X - \Pi(\eta) \in G^0\} \subset TM \oplus T^*M, \quad (68)$$

then the mechanical system with damping can be defined by

$$(\dot{x}, dH(x)) \in L, \quad x = (q, p) \in M. \quad (69)$$

The next example will illustrate classical mechanical systems with damping more concretely.

Example 6. ([22, 19]) We consider a particle moving in \mathbb{R}^3 , subject to the nonholonomic constraint $\dot{z} = y\dot{x}$ and a friction force proportional to the particle velocity. The Hamiltonian is given in terms of cartesian coordinates x, y, z and their conjugate momenta by

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2). \quad (70)$$

The LD structure L is given by the characteristic distributions

$$\begin{aligned} L \cap V &= \text{span}\left\{\frac{\partial}{\partial p_z} - y\frac{\partial}{\partial p_x}\right\} \\ \rho^*(L) &= \text{span}\{dx, dy, dz, ydp_z + dp_x, dp_y\} \end{aligned}$$

and the vector bundle map

$$\Pi = B - \tilde{R},$$

where

$$\begin{aligned} B &= \begin{bmatrix} O_3 & I_3 \\ -I_3 & O_3 \end{bmatrix}, \\ \tilde{R} &= \begin{bmatrix} O_3 & O_3 \\ O_3 & -R \end{bmatrix}, \end{aligned}$$

such that

$$R = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}$$

with $\mu_i(q) > 0$ is the directional and space-dependent damping coefficient [19]. The equations of motion in brackets is given by

$$\dot{z} = \{\{z, H\}\} = \{z, H\} + [z, H]$$

where $z = (q, p)$, or more explicitly

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_i^j, \quad (71)$$

and

$$[q_i, q_j] = [q_i, p_j] = 0, \quad [p_i, p_j] = -\delta_i^j \mu_i, \quad (72)$$

for all $i, j = 1, 2, 3$. Therefore the equations of motion read

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & -1 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & -1 & 1 & 0 & \mu_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ p_x \\ p_y \\ p_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ y \\ 0 \\ 1 \end{bmatrix} \lambda,$$

$$0 = yp_x - p_z.$$

Observe that not every point $x = (q, p) \in T^*\mathbb{R}^3$ satisfies

$$(\dot{x}, dH(x)) \in L(x), \quad (73)$$

but a proper subset

$$\chi_c = \{x \in T^*\mathbb{R}^3; dH(x) \in \rho^*(L(x))\}. \quad (74)$$

It would be interesting to study reduction of the system to a subsystem on χ_c [21].

5. Conclusions. We have defined linear and smooth Leibniz-Dirac structures as generalizations of linear and smooth Dirac structures. We have studied the geometry and dynamics of LD structures both on linear spaces and manifolds. It has been explained with several examples that LD structures are capable of formulating dissipative implicit Hamiltonian systems with constraints. We hope that LD structures will find applications in physics and related areas.

However, there remain many questions on both geometric and dynamics properties of LD structures unexplored, some of which are addressed herewith. As it is known, a more general setting of Dirac structures on Courant algebroids has a growing importance [17, 8]. Accordingly, LD structures may be extended to vector bundles such as algebroids [13, 2]. Another topic is an investigation of transformations that preserve LD structures. For this end, one can use the notions of pushed forward and pull back maps in the sense of [7]. This leads hopefully to symmetry reduction of LD structures under Lie groups.

Among LD structures, symmetric Dirac structures have the richest geometry after Dirac structures. We believe that symmetric Dirac structures are powerful tools in studying the geometry of physical systems such as gradient systems with constraints [3] and incompressible viscous fluids [19].

Acknowledgments. We are grateful to the anonymous referees for their careful reading and their helpful comments and suggestions which improved the paper essentially.

REFERENCES

- [1] R. Abraham J. E. Marsden and T. Ratiu, “Manifolds, Tensor Analysis, and Applications,” Second edition, Applied Mathematical Sciences, **75**, Springer-Verlag, New York, 1988.
- [2] P. Balseiro, M. de León, J. C. Marrero and D. Martín de Diego, *The ubiquity of the symplectic Hamiltonian equations in mechanics*, J. Geom. Mech., **1** (2009), 1–34.
- [3] G. Blankenstein, *A joined geometric structure for Hamiltonian and gradient control systems*, in “Lagrangian and Hamiltonian Methods for Nonlinear Control 2003,” IFAC, Laxenburg, (2003), 51–56.
- [4] G. Blankenstein, *Geometric modeling of nonlinear RLC circuits*, IEEE Trans. Circuits Syst. I Regul. Pap., **52** (2005), 396–404.

- [5] A. Bloch, P. S. Krishnaprasad, J. E. Marsden and T. S. Ratiu, *The Euler-Poincaré equations and double bracket dissipation*, Comm. Math. Phys., **175** (1996), 1–42.
- [6] F. Bullo and A. D. Lewis, “Geometric Control of Mechanical Systems. Modeling, Analysis, and Design for Simple Mechanical Control Systems,” Texts in Applied Mathematics, **49**, Springer-Verlag, New York, 2005.
- [7] H. Bursztyn and O. Radko, *Gauge equivalence of Dirac structures and symplectic groupoids*, Ann. Inst. Fourier (Grenoble), **53** (2003), 309–337.
- [8] H. Bursztyn, G. R. Cavalcanti and M. Gualtieri, *Reduction of Courant algebroids and generalized complex structures*, Adv. Math., **211** (2007), 726–765.
- [9] H. Cendra and S. Grillo, *Generalized nonholonomic mechanics, servomechanisms and related brackets*, J. Math. Phys., **47** (2006), 022902, 29 pp.
- [10] P. E. Crouch, *Geometric structures in systems theory*, Proceedings IEE-D, **128** (1981), 242–252.
- [11] T. J. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc., **319** (1990), 631–661.
- [12] M. Dalsmo and A. van der Schaft, *On representations and integrability of mathematical structures in energy-conserving physical systems*, SIAM J. Control Optim., **37** (1999), 54–91.
- [13] J. Grabowski and P. Urbański, *Lie algebroids and Poisson-Nijenhuis structures*, Rep. Math. Phys., **40** (1997), 195–208.
- [14] J. Grabowski and P. Urbański, *Algebroids—general differential calculi on vector bundles*, J. Geom. Phys., **31** (1999), 111–141.
- [15] M. Gualtieri, *Generalized complex geometry*, Ann. of Math. (2), **174** (2011), 75–123.
- [16] M. Jotz and T. S. Ratiu, *Dirac structures, nonholonomic systems and reduction*, Rep. Math. Phys., **69** (2012), 5–56.
- [17] Z. Liu, A. Weinstein and P. Xu, *Manin triples for Lie bialgebroids*, J. Differ. Geom., **45** (1997), 547–574.
- [18] P. J. Morrison, *A paradigm for joined Hamiltonian and dissipative systems*, Phys. D, **18** (1986), 410–419.
- [19] S. Q. H. Nguyen and L. A. Turski, *On the Dirac approach to constrained dissipative dynamics*, J. Phys. A, **34** (2001), 9281–9302.
- [20] J.-P. Ortega and V. Planas-Bielsa, *Dynamics on Leibniz manifolds*, J. Geom. Phys., **52** (2004), 1–27.
- [21] A. J. van der Schaft, *Implicit Hamiltonian systems with symmetry*, Rep. Math. Phys., **41** (1998), 203–221.
- [22] A. J. van der Schaft, “ L_2 -gain and Passivity Techniques in Nonlinear Control,” Second edition, Communications and Control Engineering Series, Springer-Verlag London, Ltd., London, 2000.
- [23] A. J. van der Schaft and B. M. Maschke, *Port-Hamiltonian systems on graphs*, SIAM J. Control Optim., **51** (2013), 906–937.

Received September 2012; revised April 2013.

E-mail address: uciftci@nku.edu.tr

Copyright of Journal of Geometric Mechanics is the property of American Institute of Mathematical Sciences and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.