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Asymptotic formulations of the eigenvalues and eigenfunctions for a boundary value problem

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In this work, a discontinuous boundary-value problem with retarded argument that contains a spectral parameter in the transmission conditions at the point of discontinuity is investigated. We obtained asymptotic formulas for the eigenvalues and eigenfunctions.

Keywords: differential equation with retarded argument; transmission conditions; asymptotics of eigenvalues and eigenfunctions

1. Introduction

Delay differential equations arise in many areas of mathematical modeling: for example, population dynamics (taking into account the gestation times), infectious diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modelling, for example, the body's reaction to CO_2, etc. in circulating blood), and chemical kinetics (such as mixing reactants), the navigational control of ships and aircraft, and more general control problems.

Boundary value problems for differential equations of the second order with retarded argument were studied in [1–7], and various physical applications of such problems can be found in [2].

In the papers [6, 7], the asymptotic formulas for the eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument and a spectral parameter in the boundary conditions were obtained.

The asymptotic formulas for the eigenvalues and eigenfunctions of the Sturm–Liouville problem with the spectral parameter in the boundary condition were obtained in [8].

The article [9] is devoted to the study of asymptotics of the solutions to the Sturm–Liouville problem with the potential and the spectral parameter having discontinuity of the first kind in the domain of definition of the solution.

In this paper, we study the eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument and spectral parameters in the transmission conditions. Namely, we consider the boundary value problem for the differential equation

$$y''(x) + q(x)y(x - \Delta(x)) + \lambda y(x) = 0 \tag{1}$$

on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$,with boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \tag{3}$$

and transmission conditions

$$y\left(\frac{\pi}{2} - 0\right) - \sqrt[3]{\lambda}\delta y\left(\frac{\pi}{2} + 0\right) = 0, \tag{4}$$

$$y'\left(\frac{\pi}{2} - 0\right) - \sqrt[3]{\lambda}\delta y'\left(\frac{\pi}{2} + 0\right) = 0, \tag{5}$$

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where the real-valued function $q(x)$ is continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $q\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} q(x)$, the real valued function $\Delta(x) \geq 0$ continuous in $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and has a finite limit $\Delta\left(\frac{\pi}{2} \pm 0\right) = \lim_{x \to \frac{\pi}{2} \pm 0} \Delta(x)$, $x - \Delta(x) \geq 0$, if $x \in \left[0, \frac{\pi}{2}\right)$; $x - \Delta(x) \geq \frac{\pi}{2}$, if $x \in \left(\frac{\pi}{2}, \pi\right]$; λ is a real spectral parameter; δ is an arbitrary real number.

It must be noted that some problems with transmission conditions that arise in mechanics (thermal condition problem for a thin laminated plate) were studied in [10].

Let $w_1(x, \lambda)$ be a solution of Equation (1) on $\left[0, \frac{\pi}{2}\right]$, satisfying the initial conditions

$$w_1(0, \lambda) = \sin\alpha, w_1'(0, \lambda) = -\cos\alpha. \tag{6}$$

The conditions (6) define a unique solution of Equation (11) on $\left[0, \frac{\pi}{2}\right]$ ([2], p. 12).

After defining the aforementioned solution, we shall define the solution $w_2(x, \lambda)$ of Equation (1) on $\left[\frac{\pi}{2}, \pi\right]$ by means of the solution $w_1(x, \lambda)$ using the initial conditions

$$w_2\left(\frac{\pi}{2}, \lambda\right) = \lambda^{-1/3}\delta^{-1} w_1\left(\frac{\pi}{2}, \lambda\right), \quad \omega_2'\left(\frac{\pi}{2}, \lambda\right) = \lambda^{-1/3}\delta^{-1}\omega_1'\left(\frac{\pi}{2}, \lambda\right). \tag{7}$$

The conditions (7) are defined as a unique solution of Equation (1) on $\left[\frac{\pi}{2}, \pi\right]$.

Consequently, the function $w(x, \lambda)$ is defined on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ by the equality

$$w(x, \lambda) = \begin{cases} \omega_1(x, \lambda), & x \in \left[0, \frac{\pi}{2}\right) \\ \omega_2(x, \lambda), & x \in \left(\frac{\pi}{2}, \pi\right] \end{cases}$$

is a solution of Equation (1) on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, which satisfies one of the boundary conditions and both transmission conditions.

Lemma 1

Let $w(x, \lambda)$ be a solution of Equation (1) and $\lambda > 0$. Then the following integral equations hold:

$$\begin{aligned} w_1(x, \lambda) = {} & \sin\alpha \cos sx - \frac{\cos\alpha}{s}\sin sx \\ & - \frac{1}{s}\int_0^x q(\tau)\sin s(x-\tau) w_1(\tau - \Delta(\tau), \lambda)\, d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right), \end{aligned} \tag{8}$$

$$\begin{aligned} w_2(x, \lambda) = {} & \frac{1}{s^{2/3}\delta} w_1\left(\frac{\pi}{2}, \lambda\right)\cos s\left(x - \frac{\pi}{2}\right) + \frac{w_1'\left(\frac{\pi}{2}, \lambda\right)}{s^{5/3}\delta}\sin s\left(x - \frac{\pi}{2}\right) \\ & - \frac{1}{s}\int_{\pi/2}^x q(\tau)\sin s(x-\tau) w_2(\tau - \Delta(\tau), \lambda)\, d\tau \quad \left(s = \sqrt{\lambda}, \lambda > 0\right). \end{aligned} \tag{9}$$

Proof

To prove this, it is enough to substitute $-s^2\omega_1(\tau, \lambda) - \omega_1''(\tau, \lambda)$ and $-s^2\omega_2(\tau, \lambda) - \omega_2''(\tau, \lambda)$ instead of $-q(\tau)\omega_1(\tau - \Delta(\tau), \lambda)$ and $-q(\tau)\omega_2(\tau - \Delta(\tau), \lambda)$ in the integrals (8) and (9), respectively and integrate by parts twice. □

Theorem 1

The problem (1)–(5) can have only simple eigenvalues.

Proof

Let $\widetilde{\lambda}$ be an eigenvalue of the problem (1)–(5) and

$$\widetilde{y}\left(x, \widetilde{\lambda}\right) = \begin{cases} \widetilde{y}_1\left(x, \widetilde{\lambda}\right), & x \in \left[0, \frac{\pi}{2}\right), \\ \widetilde{y}_2\left(x, \widetilde{\lambda}\right), & x \in \left(\frac{\pi}{2}, \pi\right] \end{cases}$$

be a corresponding eigenfunction. Then from Equations (2) and (6), it follows that the determinant

$$W\left[\widetilde{y}_1\left(0, \widetilde{\lambda}\right), w_1\left(0, \widetilde{\lambda}\right)\right] = \begin{vmatrix} \widetilde{y}_1\left(0, \widetilde{\lambda}\right) & \sin\alpha \\ \widetilde{y}_1'\left(0, \widetilde{\lambda}\right) & -\cos\alpha \end{vmatrix} = 0;$$

and by Theorem 2.2.2 in [2], the functions $\widetilde{y}_1\left(x, \widetilde{\lambda}\right)$ and $w_1\left(x, \widetilde{\lambda}\right)$ are linearly dependent on $\left[0, \frac{\pi}{2}\right]$. We can also prove that the functions $\widetilde{y}_2\left(x, \widetilde{\lambda}\right)$ and $w_2\left(x, \widetilde{\lambda}\right)$ are linearly dependent on $\left[\frac{\pi}{2}, \pi\right]$. Hence

$$\widetilde{y}_i\left(x, \widetilde{\lambda}\right) = K_i w_i\left(x, \widetilde{\lambda}\right) \quad (i = 1, 2) \tag{10}$$

for some $K_1 \neq 0$ and $K_2 \neq 0$. We must show that $K_1 = K_2$. Suppose that $K_1 \neq K_2$. From equalities (4) and (10), we have

$$\begin{aligned}\widetilde{y}\left(\frac{\pi}{2}-0,\widetilde{\lambda}\right)-\sqrt[3]{\widetilde{\lambda}}\delta\widetilde{y}\left(\frac{\pi}{2}+0,\widetilde{\lambda}\right)&=\widetilde{y_1}\left(\frac{\pi}{2},\widetilde{\lambda}\right)-\sqrt[3]{\widetilde{\lambda}}\delta\widetilde{y_2}\left(\frac{\pi}{2},\widetilde{\lambda}\right)\\&=K_1w_1\left(\frac{\pi}{2},\widetilde{\lambda}\right)-\sqrt[3]{\widetilde{\lambda}}\delta K_2w_2\left(\frac{\pi}{2},\widetilde{\lambda}\right)\\&=\sqrt[3]{\widetilde{\lambda}}\delta K_1w_2\left(\frac{\pi}{2},\widetilde{\lambda}\right)-\sqrt[3]{\widetilde{\lambda}}\delta K_2w_2\left(\frac{\pi}{2},\widetilde{\lambda}\right)\\&=\sqrt[3]{\widetilde{\lambda}}\delta(K_1-K_2)w_2\left(\frac{\pi}{2},\widetilde{\lambda}\right)=0.\end{aligned}$$

Because $\delta_1(K_1-K_2)\neq 0$, it follows that

$$w_2\left(\frac{\pi}{2},\widetilde{\lambda}\right)=0. \tag{11}$$

By the same procedure from equality (5), we can derive that

$$w_2'\left(\frac{\pi}{2},\widetilde{\lambda}\right)=0. \tag{12}$$

From the fact that $w_2\left(x,\widetilde{\lambda}\right)$ is a solution of the differential Equation (1) on $\left[\frac{\pi}{2},\pi\right]$ and satisfies the initial conditions (11) and (12), it follows that $w_2\left(x,\widetilde{\lambda}\right)=0$ identically on $\left[\frac{\pi}{2},\pi\right]$ (cf. [2, p. 12, Theorem 1.2.1]).

By using this, we may also find

$$w_1\left(\frac{\pi}{2},\widetilde{\lambda}\right)=w_1'\left(\frac{\pi}{2},\widetilde{\lambda}\right)=0.$$

From the latter discussions of $w_2\left(x,\widetilde{\lambda}\right)$, it follows that $w_1\left(x,\widetilde{\lambda}\right)=0$ identically on $\left[0,\frac{\pi}{2}\right)$. But this contradicts (6), thus completing the proof. □

2. An existance theorem

The function $\omega(x,\lambda)$ defined in Section 1 is a nontrivial solution of Equation (1) satisfying conditions (2), (4), and (5). Putting $\omega(x,\lambda)$ into Equation (3), we obtain the characteristic equation

$$F(\lambda)\equiv\omega(\pi,\lambda)\cos\beta+\omega'(\pi,\lambda)\sin\beta=0. \tag{13}$$

By Theorem 1.1, the set of eigenvalues of boundary-value problem (1)–(5) coincides with the set of real roots of Equation (13). Let $q_1=\int_0^{\pi/2}|q(\tau)|\mathrm{d}\tau$ and $q_2=\int_{\pi/2}^{\pi}|q(\tau)|\,\mathrm{d}\tau$.

Lemma 2

(1) Let $\lambda\geq 4q_1^2$. Then for the solution $w_1(x,\lambda)$ of Equation (8), the following inequality holds:

$$|w_1(x,\lambda)|\leq\frac{1}{|q_1|}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha},\quad x\in\left[0,\frac{\pi}{2}\right]. \tag{14}$$

(2) Let $\lambda\geq\max\left\{4q_1^2,4q_2^2\right\}$. Then for the solution $w_2(x,\lambda)$ of Equation (9), the following inequality holds:

$$|w_2(x,\lambda)|\leq\frac{2\sqrt[3]{2}}{\sqrt[3]{q_1^5\delta}}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha},\quad x\in\left[\frac{\pi}{2},\pi\right]. \tag{15}$$

Proof

Let $B_{1\lambda}=\max_{[0,\frac{\pi}{2}]}|w_1(x,\lambda)|$. Then from (8), it follows that, for every $\lambda>0$, the following inequality holds:

$$B_{1\lambda}\leq\sqrt{\sin^2\alpha+\frac{\cos^2\alpha}{s^2}}+\frac{1}{s}B_{1\lambda}q_1.$$

If $s\geq 2q_1$, we obtain Equation (14). Differentiating Equation (8) with respect to x, we have

$$w_1'(x,\lambda)=-s\sin\alpha\sin sx-\cos\alpha\cos sx-\int_0^x q(\tau)\cos s(x-\tau)w_1(\tau-\Delta(\tau),\lambda)\mathrm{d}\tau. \tag{16}$$

From Equations (16) and (14), it follows that, for $s \geq 2q_1$, the following inequality holds:.

$$\frac{\left|w_1'(x,\lambda)\right|}{s^{5/3}} \leq \frac{1}{\sqrt[3]{4q_1^5}}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha}. \tag{17}$$

Let $B_{2\lambda} = \max_{\left[\frac{\pi}{2},\pi\right]}|w_2(x,\lambda)|$. Then from Equations (9), (14), and (17), it follows that, for $s \geq 2q_1$ and $s \geq 2q_2$, the following inequalities hold:

$$B_{2\lambda} \leq \frac{2}{\sqrt[3]{4q_1^5}\delta}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha}+\frac{1}{2q_2}B_{2\lambda}q_2,$$

$$B_{2\lambda} \leq \frac{2\sqrt[3]{2}}{\sqrt[3]{q_1^5}\delta}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha}.$$

Hence, if $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$, we obtain Equation (15). □

Theorem 2
The problem (1)–(5) has an infinite set of positive eigenvalues.

Proof
Differentiating Equation (9) with respect to x, we obtain

$$w_2'(x,\lambda) = -\frac{\sqrt[3]{s}}{\delta}w_1\left(\frac{\pi}{2},\lambda\right)\sin s\left(x-\frac{\pi}{2}\right)+\frac{w_1'\left(\frac{\pi}{2},\lambda\right)}{\sqrt[3]{s^2}\delta}\cos s\left(x-\frac{\pi}{2}\right)-\int_{\pi/2}^{x}q(\tau)\cos s(x-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau. \quad \left(s=\sqrt{\lambda},\lambda>0\right). \tag{18}$$

From Equations (8), (9), (13), (16), and (18), we obtain

$$\begin{aligned}
&\left[\frac{1}{s^{2/3}\delta}\left(\sin\alpha\cos\frac{s\pi}{2}-\frac{\cos\alpha}{s}\sin\frac{s\pi}{2}-\frac{1}{s}\int_0^{\frac{\pi}{2}}q(\tau)\sin s\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right)\right.\\
&\quad\times\cos\frac{s\pi}{2}-\frac{1}{s^{5/3}\delta}\left(s\sin\alpha\sin\frac{s\pi}{2}+\cos\alpha\cos\frac{s\pi}{2}+\int_0^{\frac{\pi}{2}}q(\tau)\cos s\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right)\\
&\quad\left.\times\sin\frac{s\pi}{2}-\frac{1}{s}\int_{\pi/2}^{\pi}q(\tau)\sin s(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau\right]\cos\beta\\
&\quad+\left[-\frac{\sqrt[3]{s}}{\delta}\left(\sin\alpha\cos\frac{s\pi}{2}-\frac{\cos\alpha}{s}\sin\frac{s\pi}{2}-\frac{1}{s}\int_0^{\frac{\pi}{2}}q(\tau)\sin s\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right)\right.\\
&\quad\times\sin\frac{s\pi}{2}-\frac{1}{\sqrt[3]{s^2}\delta}\left(s\sin\alpha\sin\frac{s\pi}{2}+\cos\alpha\cos\frac{s\pi}{2}+\int_0^{\frac{\pi}{2}}q(\tau)\cos s\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\right)\\
&\quad\left.\times\cos\frac{s\pi}{2}-\int_{\frac{\pi}{2}}^{\pi}q(\tau)\cos s(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau\right]\sin\beta=0
\end{aligned} \tag{19}$$

There are four possible cases as follows:

1. $\sin\alpha\neq 0, \sin\beta\neq 0$;
2. $\sin\alpha\neq 0, \sin\beta = 0$;
3. $\sin\alpha = 0, \sin\beta\neq 0$;
4. $\sin\alpha = 0, \sin\beta = 0$.

In this paper, we shall only consider case 1. The other cases may be considered analogically. Let λ be sufficiently large. Then, by Equations (14) and (15), Equation (19) may be rewritten in the form

$$\sqrt[3]{s}\sin s\pi + O(1) = 0. \tag{20}$$

Obviously, for large s, Equation (20) has an infinite set of roots. Thus, the theorem is proved. □

3. Asymptotic formulas for eigenvalues and eigenfunctions

Now, we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following, we shall assume that s is sufficiently large. From Equations (8) and (14), we obtain

$$\omega_1(x,\lambda) = O(1) \quad \text{on} \quad \left[0, \frac{\pi}{2}\right]. \tag{21}$$

From Equations (9) and (15), we obtain

$$\omega_2(x,\lambda) = O(1) \quad \text{on} \quad \left[\frac{\pi}{2}, \pi\right]. \tag{22}$$

The existence and continuity of the derivatives $\omega'_{1s}(x,\lambda)$ for $0 \le x \le \frac{\pi}{2}$, $|\lambda| < \infty$, and $\omega'_{2s}(x,\lambda)$ for $\frac{\pi}{2} \le x \le \pi$, $|\lambda| < \infty$ follows from Theorem 1.4.1 in [2].

Lemma 3
In case 1

$$\omega'_{1s}(x,\lambda) = O(1), \quad x \in \left[0, \frac{\pi}{2}\right], \tag{23}$$

$$\omega'_{2s}(x,\lambda) = O(1), \quad x \in \left[\frac{\pi}{2}, \pi\right] \tag{24}$$

hold.

Proof
By differentiating Equation (8) with respect to s, we obtain, by Equation (21),

$$w'_{1s}(x,\lambda) = -\frac{1}{s}\int_0^x q(\tau)\sin s(x-\tau) w'_{1s}(\tau - \Delta(\tau),\lambda) + Z(x,\lambda), \ \ (|Z(x,\lambda)| \le Z_0). \tag{25}$$

Let $D_\lambda = \max_{[0,\frac{\pi}{2}]} \left|w'_{1s}(x,\lambda)\right|$. Then the existance of D_λ follows from continuity of derivation for $x \in \left[0, \frac{\pi}{2}\right]$. From Equation (25)

$$D_\lambda \le \frac{1}{s} q_1 D_\lambda + Z_0.$$

Now let $s \ge 2q_1$. Then $D_\lambda \le 2Z_0$ and the validity of the asymptotic formula (23) follows. Formula (24) may be proved analogically. □

Theorem 3
Let n be a natural number. For each sufficiently large n, in case 1, there is exactly one eigenvalue of the problem (1)–(5) near n^2.

Proof
We consider the expression that is denoted by $O(1)$ in Equation (20).

$$\frac{\delta}{\sin\alpha \sin\beta}\left\{-\frac{\sin(\alpha-\beta)}{s^{2/3}\delta}\cos s\pi + \frac{\cos\alpha\cos\beta}{s^{5/3}\delta}\sin s\pi + \int_0^{\frac{\pi}{2}}\left[\frac{\cos\beta}{s^{5/3}\delta}\sin s(\pi-\tau) + \frac{\sin\beta}{s^{2/3}\delta}\cos s(\pi-\tau)\right] q(\tau) w_1(\tau-\Delta(\tau),\lambda)d\tau \right.$$
$$\left. + \int_{\frac{\pi}{2}}^{\pi}\left[\frac{\cos\beta}{s}\sin s(\pi-\tau) + \sin\beta\cos s(\pi-\tau)\right] q(\tau) w_2(\tau-\Delta(\tau),\lambda)\,d\tau\right\}$$

If formulas (21)–(23) are taken into consideration, it can be shown by differentiation with respect to s that for large s this expression has bounded derivative. It is obvious that for large s the roots of Equation (20) are situated close to entire numbers. We shall show that, for large n, only one root (20) lies near to each n. We consider the function $\phi(s) = \sqrt[3]{s}\sin s\pi + O(1)$. Its derivative, which has the form $\phi'(s) = \frac{1}{3\sqrt[3]{s^2}}\sin s\pi + \sqrt[3]{s}\pi\cos\pi + O(1)$, does not vanish for s close to n for sufficiently large n. Thus, our assertion follows by Rolle's Theorem. □

Let n be sufficiently large. In what follows, we shall denote by $\lambda_n = s_n^2$ the eigenvalue of the problem (1)–(5) situated near n^2. We set $s_n = n + \delta_n$. From Equation (20), it follows that $\delta_n = O\left(\frac{1}{n^{1/3}}\right)$. Consequently,

$$s_n = n + O\left(\frac{1}{n^{1/3}}\right). \tag{26}$$

Formula (26) make it possible to obtain asymptotic expressions for eigenfunction of the problem (1)–(5). From Equations (8), (16), and (21), we obtain

$$\omega_1(x,\ \lambda) = \sin\alpha\cos sx + O\left(\frac{1}{s}\right), \tag{27}$$

$$\omega_1'(x,\ \lambda) = -s\sin\alpha\sin sx + O(1). \tag{28}$$

From Equations (9), (22), (27), and (28), we obtain

$$\omega_2(x,\ \lambda) = \frac{\sin\alpha}{s^{2/3}\delta}\cos\frac{s\pi}{2}\cos s\left(x-\frac{\pi}{2}\right) - \frac{\sin\alpha}{s^{2/3}\delta}\sin\frac{s\pi}{2}\sin s\left(x-\frac{\pi}{2}\right) + O\left(\frac{1}{s}\right)$$

$$\omega_2(x,\ \lambda) = \frac{\sin\alpha}{s^{2/3}\delta}\cos sx + O\left(\frac{1}{s}\right). \tag{29}$$

By substituting Equation (26) into Equations (27) and (29), we find that

$$u_{1n} = w_1(x,\lambda_n) = \sin\alpha\cos nx + O\left(\frac{1}{n^{1/3}}\right),$$

$$u_{2n} = w_2(x,\lambda_n) = \frac{\sin\alpha}{\delta n^{2/3}}\cos nx + O\left(\frac{1}{n}\right).$$

Hence, the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x) = \begin{cases} \sin\alpha\cos nx + O\left(\frac{1}{n^{1/3}}\right), & \text{for} \quad x\in\left[0,\frac{\pi}{2}\right), \\ \frac{\sin\alpha}{\delta n^{2/3}}\cos nx + O\left(\frac{1}{n}\right) & \text{for} \quad x\in\left(\frac{\pi}{2},\pi\right]. \end{cases}$$

Under some additional conditions, the more exact asymptotic formulas that depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

(i) The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $\left[0,\frac{\pi}{2}\right)\bigcup\left(\frac{\pi}{2},\pi\right]$ and have finite limits $q'\left(\frac{\pi}{2}\pm 0\right) = \lim_{x\to\frac{\pi}{2}\pm 0} q'(x)$ and $\Delta''\left(\frac{\pi}{2}\pm 0\right) = \lim_{x\to\frac{\pi}{2}\pm 0}\Delta''(x)$, respectively.

(ii) $\Delta'(x)\le 1$ in $\left[0,\frac{\pi}{2}\right)\bigcup\left(\frac{\pi}{2},\pi right\right]$, $\Delta(0)=0$ and $\lim_{x\to\frac{\pi}{2}+0}\Delta(x)=0$.

By using (ii), we have

$$x-\Delta(x)\ge 0,\ x\in\left[0,\frac{\pi}{2}\right) \tag{30}$$

$$x-\Delta(x)\ge\frac{\pi}{2},\ x\in\left(\frac{\pi}{2},\pi\right] \tag{31}$$

From Equations (27), (29), (30), and (31), we have

$$w_1(\tau-\Delta(\tau),\lambda) = \sin\alpha\cos s(\tau-\Delta(\tau)) + O\left(\frac{1}{s}\right), \tag{32}$$

$$w_2(\tau-\Delta(\tau),\lambda) = \frac{\sin\alpha}{s^{2/3}\delta}\cos s(\tau-\Delta(\tau)) + O\left(\frac{1}{s}\right). \tag{33}$$

Putting these expressions into Equation (19), we have

$$\begin{aligned} 0 = {} & -\frac{s^{1/3}}{\delta}\sin\alpha\sin\beta\sin s\pi + \frac{\sin(\alpha-\beta)}{s^{2/3}\delta}\cos s\pi - \frac{\sin\alpha\sin\beta}{s^{2/3}\delta} \\ & \times\left\{\cos s\pi\int_0^\pi\frac{q(\tau)}{2}\left[\cos s\Delta(\tau)+\cos s(2\tau-\Delta(\tau))\right]d\tau + \sin s\pi\int_0^\pi\frac{q(\tau)}{2}\left[\sin s\Delta(\tau)+\sin s(2\tau-\Delta(\tau))\right]d\tau\right\} + O\left(\frac{1}{s^{5/3}}\right). \end{aligned} \tag{34}$$

Let

$$K(x,s,\Delta(\tau)) = \frac{1}{2}\int_0^x q(\tau)\sin s\Delta(\tau)d\tau,$$

$$L(x,s,\Delta(\tau)) = \frac{1}{2}\int_0^x q(\tau)\cos s\Delta(\tau)d\tau. \tag{35}$$

It is obvious that these functions are bounded for $0 \le x \le \pi,\ 0 < s < +\infty$.

Under the conditions (i) and (ii), the following formulas

$$\int_0^x q(\tau)\cos s(2\tau - \Delta(\tau))d\tau = O\left(\frac{1}{s}\right), \quad \int_0^x q(\tau)\sin s(2\tau - \Delta(\tau))d\tau = O\left(\frac{1}{s}\right) \tag{36}$$

can be proved by the same technique in Lemma 3.3.3 in [2]. From Equations (34), (35), and (36), we have

$$\sin s\pi[s\sin\alpha\sin\beta + K(\pi,s,\Delta(\tau))\sin\alpha\sin\beta] - \cos s\pi[\sin\alpha\cos\beta - \cos\alpha\sin\beta - L(\pi,s,\Delta(\tau))\sin\alpha\sin\beta] + O\left(\frac{1}{s}\right) = 0.$$

Hence,

$$\tan s\pi = \frac{1}{s}[\cot\beta - \cot\alpha - L(\pi,s,\Delta(\tau))] + O\left(\frac{1}{s^2}\right).$$

Again if we take $s_n = n + \delta_n$, then

$$\tan(n+\delta_n)\pi = \tan\delta_n\pi = \frac{1}{n}[\cot\beta - \cot\alpha - L(\pi,n,\Delta(\tau))] + O\left(\frac{1}{n^2}\right);$$

hence for large n,

$$\delta_n = \frac{1}{n\pi}[\cot\beta - \cot\alpha - L(\pi,n,\Delta(\tau))] + O\left(\frac{1}{n^2}\right)$$

and finally

$$s_n = n + \frac{1}{n\pi}[\cot\beta - \cot\alpha - L(\pi,n,\Delta(\tau))] + O\left(\frac{1}{n^2}\right). \tag{37}$$

Thus, we have proven the following theorem.

Theorem 4

If conditions (i) and (ii) are satisfied, then the positive eigenvalues $\lambda_n = s_n^2$ of the problem (1)–(5) have the asymptotic representation of Equation (37) for $n \to \infty$.

We now may obtain a sharper asymptotic formula for the eigenfunctions. From Equations (8) and (32)

$$w_1(x,\lambda) = \sin\alpha\cos sx - \frac{\cos\alpha}{s}\sin sx - \frac{\sin\alpha}{s}\int_0^x q(\tau)\sin s(x-\tau)\cos s(\tau - \Delta(\tau))d\tau + O\left(\frac{1}{s^2}\right).$$

Thus, from Equations (35) and (36)

$$w_1(x,\lambda) = \sin\alpha\cos sx\left[1 + \frac{K(x,\ s,\ \Delta(\tau))}{s}\right] - \frac{\sin sx}{s}[\cos\alpha + \sin\alpha L(x,\ s,\ \Delta(\tau))] + O\left(\frac{1}{s^2}\right). \tag{38}$$

Replacing s by s_n and using Equation (37), we have

$$\begin{aligned} u_{1n}(x) &= w_1(x,\lambda_n) \\ &= \sin\alpha\left\{\cos nx\left[1 + \frac{K(x,n,\Delta(\tau))}{n}\right] - \frac{\sin nx}{n\pi}[(\cot\beta - \cot\alpha - L(\pi,n,\Delta(\tau)))x + (\cot\alpha + L(x,n,\Delta(\tau)))\pi]\right\} + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{39}$$

From Equations (16), (32), and (35), we have

$$\begin{aligned}\frac{w_1'(x,\lambda)}{s^{5/3}} &= -\frac{\sin\alpha\sin sx}{s^{2/3}}\left(1+\frac{K(x,\ s,\ \Delta(\tau))}{s}\right)\\ &\quad-\frac{\cos sx}{s^{5/3}}(\cos\alpha+\sin\alpha L(x,\ s,\ \Delta(\tau)))+O\left(\frac{1}{s^2}\right),x\in\left(0,\frac{\pi}{2}\right].\end{aligned} \tag{40}$$

From Equations (9), (33), (36), (38), and (40), we have

$$\begin{aligned}w_2(x,\lambda) &= \frac{1}{s^{2/3}\delta}\left\{\sin\alpha\cos\frac{s\pi}{2}\left[1+\frac{K\left(\frac{\pi}{2},s,\Delta(\tau)\right)}{s}\right]-\frac{\sin\frac{s\pi}{2}}{s}\left[\cos\alpha+\sin\alpha L\left(\frac{\pi}{2},s,\Delta(\tau)\right)\right]+O\left(\frac{1}{s^2}\right)\right\}\cos s\left(x-\frac{\pi}{2}\right)\\ &\quad-\frac{1}{\delta}\left\{\sin\alpha\sin\frac{s\pi}{2}\left[1+\frac{K\left(\frac{\pi}{2},s,\Delta(\tau)\right)}{s}\right]-\frac{\cos\frac{s\pi}{2}}{s}\left[\cos\alpha+\sin\alpha L\left(\frac{\pi}{2},s,\Delta(\tau)\right)\right]+O\left(\frac{1}{s^2}\right)\right\}\sin s\left(x-\frac{\pi}{2}\right)\\ &\quad-\frac{1}{s}\int_{\pi/2}^{x}q(\tau)\sin s(x-\tau)\left[\frac{\sin\alpha}{s^{2/3}\delta}\cos s(\tau-\Delta(\tau))+O\left(\frac{1}{s}\right)\right]d\tau\\ &= \frac{\sin\alpha}{s^{2/3}\delta}\cos sx\left[1+\frac{K\left(\frac{\pi}{2},s,\Delta(\tau)\right)}{s}\right]-\frac{\sin sx}{s^{5/3}\delta}\left(\cos\alpha+\sin\alpha L\left(\frac{\pi}{2},s,\Delta(\tau)\right)\right)\\ &\quad-\frac{\sin\alpha}{s^{5/3}\delta}\int_{\pi/2}^{x}\frac{q(\tau)}{2}\left[\sin s(x-\Delta(\tau))+\sin s\left(x-(2\tau-\Delta(\tau))\right)\right]d\tau+O\left(\frac{1}{s^2}\right)\\ &= \frac{\sin\alpha}{s^{2/3}\delta}\cos sx\left[1+\frac{K(x,s,\Delta(\tau))}{s}\right]-\frac{\sin sx}{s^{5/3}\delta}(\cos\alpha+\sin\alpha L(x,s,\Delta(\tau)))+O\left(\frac{1}{s^2}\right),\ x\in\left(\frac{\pi}{2},\pi\right].\end{aligned}$$

Now, replacing s by s_n and using Equation (37), we have

$$u_{2n}(x)=\frac{\sin\alpha}{n^{2/3}\delta}\left\{\cos nx\left[1+\frac{K(x,n,\Delta(\tau))}{n}\right]-\frac{\sin nx}{n^{5/3}\pi}\times[(\cot\beta-\cot\alpha-L(\pi,n,\Delta(\tau)))x+(\cot\alpha+L(x,n,\Delta(\tau)))\pi]\right\}+O\left(\frac{1}{n^2}\right). \tag{41}$$

Thus, we have proven the following theorem.

Theorem 5

If conditions (i) and (ii) are satisfied then, the eigenfunctions $u_n(x)$ of the problem (1)–(5) have the following asymptotic representation for $n\to\infty$:

$$u_n(x)=\begin{cases}u_{1n}(x) & \text{for}\quad x\in\left[0,\frac{\pi}{2}\right)\\ u_{2n}(x) & \text{for}\quad x\in\left(\frac{\pi}{2},\pi\right],\end{cases}$$

where $u_{1n}(x)$ and $u_{2n}(x)$ are defined as in Equations (39) and (41), respectively.

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