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Energy Consideration of a Capacitor Modelled Using Conformal Fractional-Order Derivative

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Abstract

Fractional order circuit elements have become important parts of electronic circuits to model systems including supercapacitors, filters, and many more. The conformal fractional derivative (CFD), which is a new basic fractional derivative, has been recently used to model supercapacitors successfully. It is essential to know how electronic components behave under excitation with different types of voltage and current sources. A CFD capacitor is not a well-known element and its usage in circuits is barely examined in the literature. In this research, it is examined how to calculate the stored energy of a CFD capacitor with a series resistor supplied from a DC voltage source. The solutions given in this study may be used in circuits where supercapacitors are used.

1. Introduction

Fractional derivative (FD) first appeared in a note which was written to L'Hospital in 1695 [1]. In the last decades, the application of fractional calculus has attracted the attention of many fields of science thanks to its applicability in many subjects [2-3]. The electrical transmission line analysis circa 1890 was described by using fractional derivate operators by Oliver Heaviside [4]. The fractional-order circuits are suitable elements to model different types of elements including capacitors, inductors and memristors [5-10]. Moreover, the fractional-order circuits elements can be used to model or design all sorts of filters, controllers and oscillators [7-8, 11-15]. The new simple fractional derivative is called "the conformable fractional derivative (CFD)" depending on the familiar limit definition of the derivative of a function and that breaks with other definitions in 2014 [16]. This theory is analysed and improved with some approaches in [17-18]. Nonetheless, a CFD is simply not a fractional derivative; it is clearly a first-order derivative which is multiplied by an additional factor depending on the independent variable. This new description has the benefits of being dissimilar from other types of fractional differentials and can be thought of as a natural extension of the classical one. It is usable and suitable for many enlargements to classical use of calculus, such as Taylor power series extension, the mean value problem, the product of two functions, and many other fields in Math. There is an obvious difference between the Riemann-Liouville fractional derivative of which a constant is not zero and the conformal fractional derivative of which a constant equal zero. Because of this property, the conformal fractional derivative has become an interesting topic and a hot study area for researchers. The conformal derivative has also outweighed the other types of fractional derivatives when compared with them due to its simpleness and showing similar performance [18]. Fractional RC and LC electrical circuits have been examined in [19]. Along

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with other electrical circuit elements, supercapacitors have also been modelled with fractional circuit elements in [20-22]. Caputo and CFD fractional-order derivatives are applied in the analysis of the fractional electrical circuits fed with a sinusoidal signal [23-24]. Analytical solutions of electrical circuits modelled with CFD are given in [25]. Electrical circuits described by fractional conformable derivatives have been examined in [25]. An electric circuit containing a supercapacitor modelled with the CFD has been inspected in [26].

The stored energy in an LTI capacitor is formulated in the essential course books. Furthermore, energy efficiency is an important area in many fields of electronics. Studies of the circuits with LTI capacitors and their efficiency are examined in the literature [27-28]. A capacitor is designed for storing energy; therefore, energy consideration is one of the most important topics for all kinds of capacitors such as LTIs, supercapacitors, CFD capacitors, etc. [29]. The energy of the CFD capacitor is considered an interesting research area [26]. The energy of the capacitors modeled with FD is inspected in [29-30] to find usable results. To the best of our knowledge, a method to calculate the energy of the CFD capacitors is not given in the literature yet. In this study, first, it is shown that just a voltage-dependent energy formula for the CFD capacitors is not possible due to the time dependence, and then, it is shown that using the CFD voltage and current as a function of time obtained from the circuit analysis, the energy stored in a CFD capacitor can be found. As an example, the energy stored in a CFD capacitor connected to a series resistor supplied from a constant source has been calculated and its charging energy efficiency is examined. The required waveforms are plotted with Matlab™. Energy and the energy efficiency of the circuit are examined for different α values to provide insight into how a CFD capacitor gives different responses with respect to time. Its charging energy efficiency has been compared to that of an LTI capacitor.

The remainder of the paper is organized as follows. In the second section, the CFD and the CFD capacitor model are presented. In the third section, an LTI capacitor circuit with a series resistor by a DC supply and its stored energy is reviewed. In the fourth section, it is shown that it is not possible to find a generic stored energy formula for a CFD capacitor. In the fifth section, the charging of a CFD capacitor with a series resistor supplied from a constant voltage source is examined and its stored energy is found using the incomplete gamma function. In the sixth section, the circuit is simulated, and the energy of all circuit elements and their charging efficiency are calculated. The paper is finished with the conclusion section.

2. The Conformal Fractional Derivative and the CFD Capacitor Constitutional Law

The Conformal Fractional Derivative (CFD), which is introduced in [16], for $0<\alpha\leq 1$ and $t\geq 0$, it is described as :

$$D_\alpha f(t)=\frac{d^\alpha f(t)}{dt^\alpha}=f^{'}(t)t^{1-\alpha}=\frac{df(t)}{dt}t^{1-\alpha} \tag{1}$$

More information about the CFD can be found in [16-18]. In literature, it has been shown that the fractional derivate can be used to model supercapacitors [26, 31-33]. The CFD capacitor constitutive law given in [29] can be expressed as follows:

$$i_{c_\alpha}(t)=C_\alpha\,\frac{d^\alpha v_C(t)}{dt^\alpha} \tag{2}$$

where $i_C(t), v_C(t)$ and C_α are current, voltage and capacitor coefficient of the CFD capacitor respectively.

Eq. (2) is used to model the CFD capacitor throughout the rest of the paper.

3. The Stored Energy in an LTI Capacitor and the Energy Efficiency of such Circuit when Charged Throughout a Series Resistor by a DC Voltage Source

The circuit of an LTI capacitor is shown in Figure 1 when the LTI capacitor is connected in series with an LTI resistor supplied by a DC input signal. The fundamental solution of circuit is well-known and can be found in most of the fundamentals of the circuit theory books [34-36]. Also, the current, voltage and energy of the LTI capacitor are given as:

$$i_C(t)=\frac{V_{DC}e^{-t/\tau}}{R_s} \tag{3}$$

$$V_C(t)=V_{DC}\left(1-e^{-t/\tau}\right) \tag{4}$$

$$E_C=\frac{C_sV_c^2}{2} \tag{5}$$

If the LTI capacitor is not initially charged, its voltage rises from zero to V_{DC}. Total energy loss in the resistor and the energy radiated into the medium is simply equal to the energy stored in the LTI capacitor. In [37], it is shown that the total energy needed to charge the LTI capacitor does not depend on the resistance of the circuit shown in Figure 1. The energy value would not change even though the resistance of the series resistor was time-dependent [37]. The energy efficiency of the circuit is equal to fifty percent.

When the capacitor is charged by a constant voltage source, the same amount of energy is dissipated.

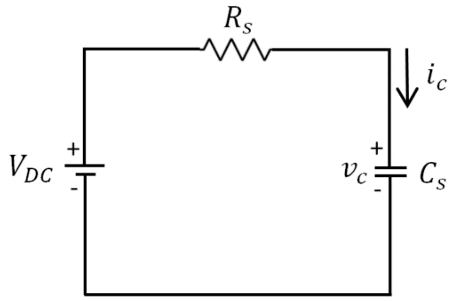

Figure 1. An LTI Capacitor supplied a constant voltage source

4. The Energy Formula of a CFD Capacitor

In this part of the study, the stored energy of a CFD capacitor is shown to be time dependent. Using Eq. (2), the CFD capacitor power is written as

$$P_{c_\alpha} = V_{c_\alpha}(t) i_{c_\alpha}(t) = V_{c_\alpha}(t) C_\alpha \frac{d^\alpha v_C(t)}{dt^\alpha} = C_\alpha V_{c_\alpha}(t) \frac{dv_{c_\alpha}(t)}{dt} t^{1-\alpha} \quad (6)$$

Then, the energy of the CFD capacitor could be described as

$$\begin{aligned} E_{c_\alpha}(t) &= \int_{t=0}^{t} P_{c_\alpha}(t)dt = \int_{t=0}^{t} V_{c_\alpha}(t) C_\alpha \frac{d^\alpha v_C(t)}{dt^\alpha} dt \\ E_{c_\alpha} &= C_\alpha \int_{t=0}^{t} V_{c_\alpha}(t) \frac{dv_{c_\alpha}(t)}{dt} t^{1-\alpha} dt \end{aligned} \quad (7)$$

The Eq. (7) is time dependent. That's why a general energy formula just depending on the capacitor voltage like Eq. (5) cannot be obtained except for $\alpha = 1$. However, if the CFD voltage as a function of time is known, it might be possible to obtain an energy formula for the circuit where it is placed. The integral may still be not solvable for some input waveforms. In this case, numerical integration methods such as the trapezoidal rule can be employed to calculate the stored energy in the CFD capacitor.

5. The Energy Calculation of a CFD Capacitor

When the LTI capacitor in Figure 1 is replaced with a CFD capacitor, the circuit shown in Figure 2 is obtained. For $t \geq 0$, the circuit is described with the following differential equation:

$$V_{DC} = i_{c_\alpha}(t) R_s + v_{c_\alpha}(t) \quad (8)$$

Using Eq. (2), the differential equation is arranged as

$$V_{DC} = R_s C_\alpha \frac{dv_{c_\alpha}(t)}{dt} t^{1-\alpha} + v_{c_\alpha}(t) \quad (9)$$

By arranging both sides of equation, it is turned into a first order differential equation;

$$\frac{dv_{c_\alpha}(t)}{dt} + \frac{t^{\alpha-1}}{R_s C_\alpha} v_{c_\alpha}(t) = \frac{V_{DC}}{R_s C_\alpha} t^{\alpha-1} \quad (10)$$

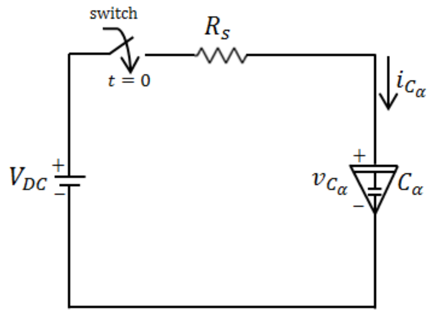

Figure 2. The CFD capacitor connected to a series resistor is fed by a constant voltage source

For a first order differential equation, whose form is of $v_{c_\alpha}(t)' + p(t) v_{c_\alpha}(t) = q(t)$, an integral factor can be used to solve the equation [38]. The both side of equation is multiplied with the integrating factor μ to find the function $v_{c_\alpha}(t)$:

$$\mu = e^{\int p(t)dt} \rightarrow \mu(v_{c_\alpha}(t)' + p(t) v_{c_\alpha}(t)) = \mu q(t) \qquad (\mu v_{c_\alpha}(t))' = \mu q(t) \quad (11)$$

$$\int (\mu v_{c_\alpha}(t))' dt = \int \mu q(t) dt \rightarrow \mu v_{c_\alpha}(t) = \int \mu q(t) dt \quad (12)$$

The integration factor is found as

$$\mu = e^{\int \frac{t^{\alpha-1}}{R_s C_\alpha} dt} = e^{\frac{t^\alpha}{R_s C_\alpha \alpha}} \quad (13)$$

The CFD capacitor voltage now can be written as

$$\begin{aligned} v_{c_\alpha}(t) &= \frac{1}{\mu} \int \mu q(t) dt = e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} \int e^{\frac{t^\alpha}{R_s C_\alpha \alpha}} \frac{V_{DC}}{R_s C_\alpha} t^{\alpha-1} dt \\ v_{c_\alpha}(t) &= V_{DC} e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} \int \frac{1}{R_s C_\alpha} e^{\frac{t^\alpha}{R_s C_\alpha \alpha}} t^{\alpha-1} dt \end{aligned} \quad (14)$$

Making the following substitution: $u = e^{\frac{t^\alpha}{R_s C_\alpha \alpha}} \rightarrow du = \frac{1}{R_s C_\alpha} e^{\frac{t^\alpha}{R_s C_\alpha \alpha}} t^{\alpha-1} dt$, the Eq. (14) is converted into;

$$v_{c_a}(t) = V_{DC} e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} \int du$$
$$v_{c_a}(t) = V_{DC} e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} \left(u + K\right) \quad (15)$$

When reverse substitution is performed, the CFD capacitor voltage is found as

$$v_{c_a}(t) = V_{DC} e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} (e^{\frac{t^\alpha}{R_s C_\alpha \alpha}} + K)$$
$$v_{c_a}(t) = V_{DC} + V_{DC} e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} K \quad (16)$$

If the initial condition at t=0 is applied, the integration constant K is found;

$$v_{c_a}(0) = V_{DC} + V_{DC} K \rightarrow K = \frac{v_{c_a}(0)}{V_{DC}} - 1 \quad (17)$$

The equation is rearranged after finding K, the $v_{c_a}(t)$ is written as

$$v_{c_a}(t) = V_{DC} + v_{c_a}(0) e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} - V_{DC} e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}}$$
$$v_{c_a}(t) = V_{DC}(1 - e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}}) + v_{c_a}(0) e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} \quad (18)$$

After $v_{c_a}(t)$ is found, the CFD capacitor current is written with using the terminal equation. Then, it is described as

$$i_{c_\alpha} = C_\alpha t^{1-\alpha} \frac{d}{dt} \left(V_{DC} + e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} (v_{c_a}(0) - V_{DC}) \right)$$
$$= \frac{1}{R_s} e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} (V_{DC} - v_{c_a}(0)) \quad (19)$$

For the circuit given in Figure 2, both the CFD voltage and current are obtained. Then, the power of CFD the capacitor is obtained as

$$p(t) = i_{c_\alpha}(t) v_{c_\alpha}(t) \quad (20)$$

$$p(t) = \frac{1}{R_s} e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} (V_{DC} - v_{c_a}(0))$$
$$\left(V_{DC} + e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} (v_{c_a}(0) - V_{DC}) \right) \quad (21)$$

$$p(t) = e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} \left(-\frac{V_{DC}}{R_s} v_{c_a}(0) + \frac{V_{DC}{}^2}{R_s} \right)$$
$$+ e^{-\frac{2t^\alpha}{R_s C_\alpha \alpha}} \left(-\frac{1}{R_s} v_{c_a}{}^2(0) + \frac{2V_{DC}}{R_s} v_{c_a}(0) - \frac{V_{DC}{}^2}{R_s} \right) \quad (22)$$

The power can be used to come up with a solution for the energy stored within the CFD capacitor which is designated as *E(t)*. If the integral is divided into two parts, the equation of energy is written as

$$E(t) = \int_0^t p(t) dt = \int_0^t E_1(t) dt + \int_0^t E_2(t) dt \quad (23)$$

$$E_1(t) = \int_0^t e^{-\frac{t^\alpha}{R_s C_\alpha \alpha}} \left(-\frac{V_{DC}}{R_s} v_{c_a}(0) + \frac{V_{DC}{}^2}{R_s} \right) dt \quad (24)$$

$$E_2(t) = \int_0^t e^{-\frac{2t^\alpha}{R_s C_\alpha \alpha}} \left(-\frac{1}{R_s} v_{c_a}{}^2(0) + \frac{2V_{DC}}{R_s} v_{c_a}(0) - \frac{V_{DC}{}^2}{R_s} \right) dt \quad (25)$$

Eq. (25) can be simplified using a special integral function called the incomplete gamma function [39]. The gamma function $\Gamma(s)$ can be described to the incomplete gamma function $\Gamma(s,x)$ such that $\Gamma(a) = \Gamma(s,0)$. As a result of this, the upper incomplete gamma function is given by

$$\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} dt \quad (26)$$

When the substitution $u_1 = \frac{t}{\left(R_s C_\alpha \alpha\right)^{1/\alpha}}$ is applied in $E_1(t)$ integral, $\frac{du_1}{dt} = \frac{1}{\left(R_s C_\alpha \alpha\right)^{1/\alpha}} \rightarrow dt = \left(R_s C_\alpha \alpha\right)^{1/\alpha} du_1$ is found and the equation is converted into the incomplete gamma function.

$$E_1(t) = \left(-\frac{V_{DC}}{R_s} v_{c_a}(0) + \frac{V_{DC}{}^2}{R_s} \right) \left(R_s C_\alpha \alpha\right)^{1/\alpha} \int e^{-u_1{}^\alpha} du_1$$
$$= \left(-\frac{V_{DC}}{R_s} v_{c_a}(0) + \frac{V_{DC}{}^2}{R_s} \right) \left(R_s C_\alpha \alpha\right)^{1/\alpha} \left(-\frac{\Gamma(\frac{1}{\alpha}, u_1{}^\alpha)}{a}\right) \quad (27)$$

If the substitution is reversed, $E_1(t)$ is found as

$$E_1(t) = -\frac{\Gamma(\frac{1}{\alpha}, \frac{t^{\alpha}}{R_s C_{\alpha} \alpha})t}{a(\frac{t^{\alpha}}{R_s C_{\alpha} \alpha})^{1/\alpha}} \left(-\frac{V_{DC}}{R_s} v_{c_\alpha}(0) + \frac{V_{DC}^2}{R_s} \right) \quad (28)$$

Similarly, $E_2(t)$ can be simplified solved with using same method. When the substitution $u_2 = \frac{2^{1/\alpha} t}{\left(R_s C_{\alpha} \alpha\right)^{1/\alpha}}$ is used, $\frac{du_2}{dt} = \frac{2^{1/\alpha}}{\left(R_s C_{\alpha} \alpha\right)^{1/\alpha}} \rightarrow dt = \frac{\left(R_s C_{\alpha} \alpha\right)^{1/\alpha}}{2^{1/\alpha}} du_2$ is written and, using the incomplete gamma function, the integral is found as

$$\begin{aligned} E_2(t) &= \left(-\frac{1}{R_s} v_{c_\alpha}{}^2(0) + \frac{2V_{DC}}{R_s} v_{c_\alpha}(0) - \frac{V_{DC}{}^2}{R_s} \right) \frac{\left(R_s C_{\alpha} \alpha\right)^{1/\alpha}}{2^{1/\alpha}} \int e^{-u_2{}^{\alpha}} du_2 \\ &= \left(-\frac{1}{R_s} v_{c_\alpha}{}^2(0) + \frac{2V_{DC}}{R_s} v_{c_\alpha}(0) - \frac{V_{DC}{}^2}{R_s} \right) \frac{\left(R_s C_{\alpha} \alpha\right)^{1/\alpha}}{2^{1/\alpha}} \left(-\frac{\Gamma(\frac{1}{\alpha}, u_2{}^{\alpha})}{a}\right) \end{aligned} \quad (29)$$

When reverse substitution is performed, $E_2(t)$ is found as

$$E_2(t) = -\frac{\Gamma(\frac{1}{\alpha}, \frac{2t^{\alpha}}{R_s C_{\alpha} \alpha})t}{a 2^{1/\alpha} (\frac{t^{\alpha}}{R_s C_{\alpha} \alpha})^{1/\alpha}} \left(-\frac{1}{R_s} v_{c_\alpha}{}^2(0) + \frac{2V_{DC}}{R_s} v_{c_\alpha}(0) - \frac{V_{DC}{}^2}{R_s} \right) \quad (30)$$

Then $E(t) = \int_0^t E_1(t)dt + \int_0^t E_2(t)dt$ is applied, the energy equation of the CFD capacitor is written as

$$\begin{aligned} E(t) = &-\frac{\Gamma\left(\frac{1}{\alpha}, \frac{t^{\alpha}}{R_s C_{\alpha} \alpha}\right)t}{\alpha(\frac{t^{\alpha}}{R_s C_{\alpha} \alpha})^{1/\alpha}} \left(-\frac{V_{DC}}{R_s} v_{c_\alpha}(0) + \frac{V_{DC}{}^2}{R_s} \right) \\ &-\frac{\Gamma\left(\frac{1}{\alpha}, \frac{2t^{\alpha}}{R_s C_{\alpha} \alpha}\right)t}{\alpha 2^{\frac{1}{\alpha}} (\frac{t^{\alpha}}{R_s C_{\alpha} \alpha})^{1/\alpha}} \left(-\frac{1}{R_s} v_{c_\alpha}{}^2(0) + \frac{2V_{DC}}{R_s} v_{c_\alpha}(0) - \frac{V_{DC}{}^2}{R_s} \right) \end{aligned} \quad (31)$$

The energy loss within the equivalent series resistor is written as

$$\begin{aligned} E_{LOSS} &= \int_{t=0}^{t=T_{last}} i_{c_\alpha}{}^2 R_s dt == \int_{t=0}^{t=T_{last}} \left(\frac{1}{R_s} e^{-\frac{t^{\alpha}}{R_s C_{\alpha} \alpha}} (V_{DC} - v_{c_\alpha}(0)) \right)^2 R_s dt \\ E_{LOSS} &= \int_{t=0}^{t=T_{last}} \frac{1}{R_s} e^{\frac{-2t^{\alpha}}{R_s C_{\alpha} \alpha}} (V_{DC} - v_{c_\alpha}(0))^2 dt \end{aligned} \quad (32)$$

When the circuit is supplied by DC signal, the power supplied by the source is calculated as

$$p_s(t) = V_{DC} i_{c_\alpha} \quad (33)$$

The energy supplied by the source is given as

$$E_s(t) = \int_0^t V_{DC} i_{c_\alpha} = \int_0^t V_{DC} \frac{1}{R_s} e^{-\frac{t^{\alpha}}{R_s C_{\alpha} \alpha}} (V_{DC} - v_{c_\alpha}(0)) dt \quad (34)$$

$$E_s(t) = \frac{V_{DC}}{R_s} (V_{DC} - v_{c_\alpha}(0)) \int_0^t e^{-\frac{t^{\alpha}}{R_s C_{\alpha} \alpha}} dt \quad (35)$$

Using the incomplete gamma funcion, Eq. 35 can be simplied and written as

$$E_s(t) = -\frac{\Gamma(\frac{1}{\alpha}, \frac{t^{\alpha}}{R_s C_{\alpha} \alpha})t}{a(\frac{t^{\alpha}}{R_s C_{\alpha} \alpha})^{1/\alpha}} \left(\frac{V_{DC}}{R_s} (V_{DC} - v_{c_\alpha}(0)) \right) \quad (36)$$

Also, it can be written using charge equation and it is described as

$$E_S = \int_{t=0}^{t=T_{last}} V_{DC} dq_C = V_{DC} \int_{t=0}^{t=T_{last}} dq_C = V_{DC} \left(q\left(T_{last}\right) - q\left(0\right) \right) \quad (37)$$

The energy efficiency of the charging process is calculated as

$$\eta = \frac{E_{STORED}}{E_S} = \frac{E_C}{E_S} \quad (38)$$

6. Simulations and Results

In this section, the voltage and current of the CFD capacitor for three different α values calculated when the circuit is fed by the constant voltage source are shown in Figures 3 and 4, respectively. Moreover, $V_{DC} = 5\ V$, $v_{c_\alpha}(0) = 0\ V/s^{1-\alpha}$, $R_s = 2.5\ \Omega$ and $C_\alpha = 1\ F/s^{1-\alpha}$ are applied in the CFD capacitor equations.

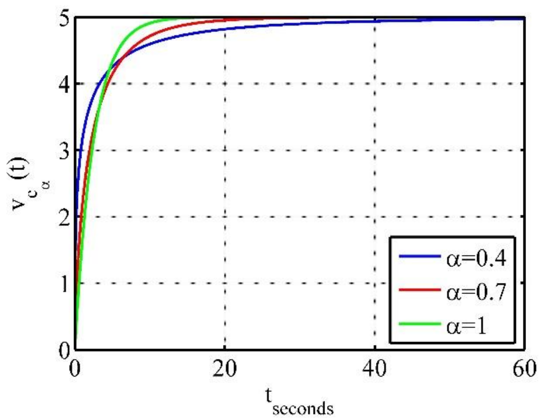

Figure 3. The voltage of the CFD capacitor for three different α values

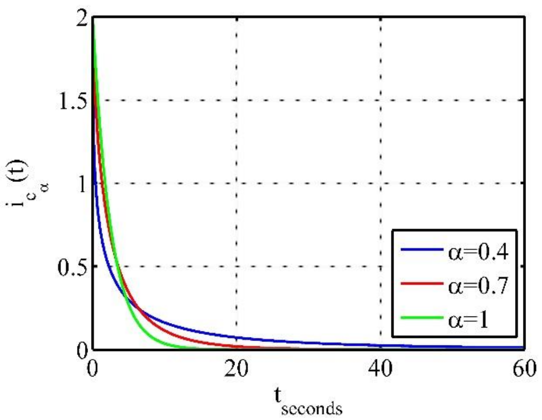

Figure 4. The current of the CFD capacitor for three different α values

The stored energy of the CFD capacitor, the supplied energy by the source, and the CFD capacitor charging energy efficiency are simulated for three different α values as shown in Figures 5-7. A Matlab code has been written to calculate the energy of the CFD capacitor. Furthermore, $V_{DC} = 5\ V$, $v_{c_\alpha}(0) = 0\ V/s^{1-\alpha}$, $R_s = 2.5\ \Omega$ and $C_\alpha = 1\ F/s^{1-\alpha}$ are used in the equations.

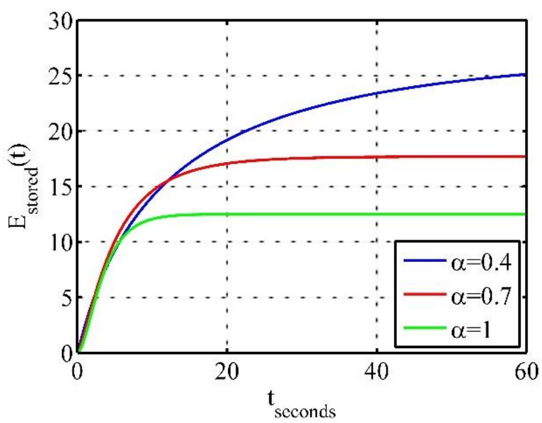

Figure 5. The energy of the CFD capacitor for three different α values

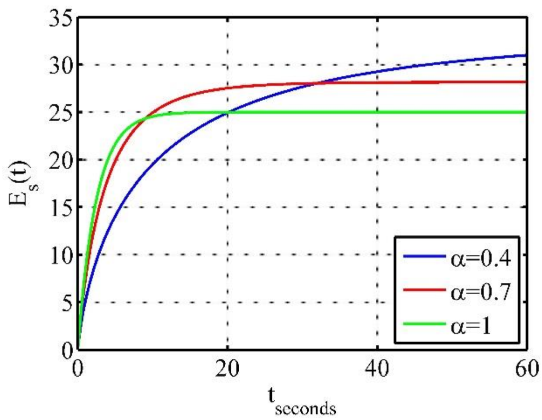

Figure 6. The supplied energy for three different α values.

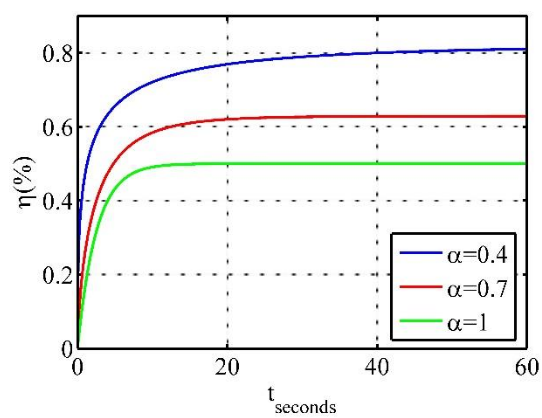

Figure 7. The energy efficiency for three different α values.

The graphs of energy loss within the series resistor for three different resistor values are sketched in Figures 8-10 when $\alpha = 0.4$, $\alpha = 0.7$, and $\alpha = 1$ are applied, respectively. Moreover, $V_{DC} = 5\ V$, $v_{c_\alpha}(0) = 0\ V/s^{1-\alpha}$, and $C_\alpha = 1\ F/s^{1-\alpha}$ are used in the equations. According to Figures 8-10, the energy loss within the resistor does depend on its resistance except for the case $\alpha = 1$.

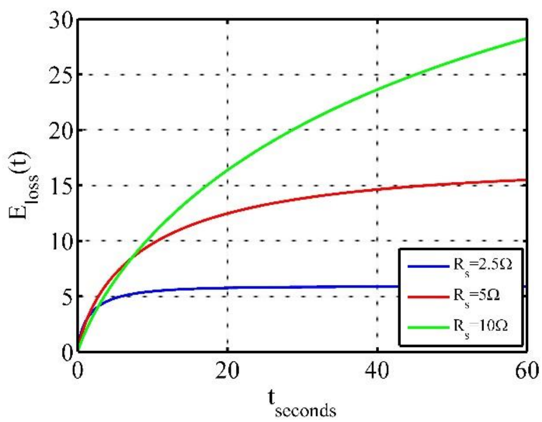

Figure 8. The energy loss within the series resistor for three different resistor values for $\alpha = 0.4$

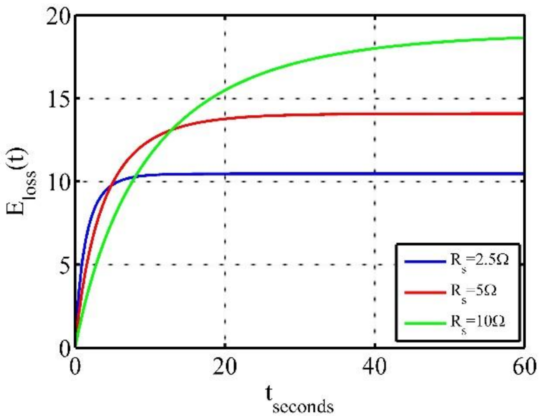

Figure 9. The energy loss within the series resistor for three different resistor values for $\alpha = 0.7$.

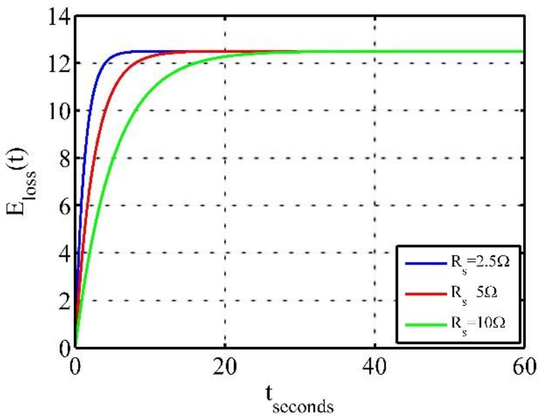

Figure 10. The energy loss within the series resistor for three different resistor values for $\alpha = 1$.

7. Conclusion

Supercapacitors or ultra-capacitors can be modelled with fractional-order derivatives. If a supercapacitor is modelled with the CFD, its energy cannot be calculated as easily as it can be done for an LTI capacitor. Therefore, it should be examined how to calculate the energy stored in a CFD capacitor. In this study, a method is given to calculate the energy of a CFD capacitor if it is charged through a resistor from a DC supply. The solution of its energy has been found as a special function called the incomplete gamma function. The energy equations in the circuit are simulated using Matlab. The charging efficiency of the CFD capacitor is shown to be dependent on the parameter and the series resistor resistance. It is shown that it can be higher than 50%, which the value is obtained for the LTI capacitor case. It is imperative to comprehend that the energy formula given here could not be used for different sources and different circuits. The CFD capacitor is a new circuit element and its analysis with the other circuit elements combined with different waveforms should be made. After the analysis of each new circuit topology, the energy of the CFD capacitor can be found using the method given here. For each case, a different energy formula would be obtained. Such analysis detailed in the paper may pave the way to understanding the CFD capacitor better, make its usage easier, and find new usage areas.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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