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Pedal Sets of Unitals in Projective Planes of Order 16

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Abstract

In this article, we perform computer searches for pedal sets of all known unitals in the known planes of order 16. Special points of unitals having at least one special tangent are studied in detail. It is shown that unitals without special points exist. Open problems regarding the computational results presented in this study are discussed. A conjecture about the numbers of line types of a unital U and its dual unital $U^{\perp}$ is formulated.

1. Introduction

We assume familiarity with the basic facts from finite geometries and combinatorial design theory [1]- [3].
A t-(v,k,λ) design is a pair D=$\{X,B\}$ of a set X of cardinality v, called points, and a collection B of k-subsets of X, called blocks, such that every t points appear together in exactly λ blocks. A parallel class of a design D is a collection of blocks that partitions the point set of D. A resolution of D is a partition of the collection of blocks of D into disjoint parallel classes. A design D is resolvable if it has at least one resolution.
Let π be a projective plane of order q^2. A *unital* embedded in π is defined to be a set U of q^3+1 points of π meeting lines of the plane in either *one* point or $q+1$ points. The sets of the intersections of the lines of π with U at $q+1$ points form a 2-$(q^3+1, q+1, 1)$ design.
A classical example of a unital is the Hermitian unital $H(q)$ defined by the absolute points of a unitary polarity in $PG(2,q^2)$. In 1976, Buekenhout provided two methods for constructing unitals [4]. In 1979, Metz used one of Buekenhout's method to construct a non-classical unital in a Desarguesian plane of order q^2 [5], and in 1994, Barwick showed that any unitals constructed by the other method of Buekenhout is a classical unital [6]. In 1988, for every odd prime power q, Rosati constructed a unital in Hughes planes of order q [7], and in 1990, Kestenband generalized Rosati's construction [8]. Some other studies of unitals can be found in [9]- [12].
There exist q^3+1 lines meeting a unital U at one point, called *tangent lines* to U, and $q^2(q^2-q+1)$ lines meeting U at $q+1$ points, called *secant lines* to U. For any point $P \notin U$, the number of tangents and secants through P are $q+1$ and q^2-q, respectively [1]. The set of the $q+1$ intersections of tangents through P with U is called the *pedal set* of P. P is called a *special point* if its pedal set is collinear. A *special tangent* is defined to be a tangent having q^2 special points.
In this study, pedal sets of all known unitals in the known projective planes of order 16 are computed. Special points of unitals having at least one special tangent are studied in detail. It is shown that unitals without special points exist. Details of the numbers of pedal sets for each possible line type are reported.
Through the paper, a line with p points will be denoted by p-line.

2. Pedal sets of unitals in planes of order 16

Twenty-two projective planes of order 16 are known to exist. The names of the planes are in accordance with [13]: $PG(2,16)$, BBH1, SEMI2, SEMI4, BBH2, BBS4, DEMP, DSFP, HALL, LMRH, MATH, JOHN, and JOWK. Specific line sets of the

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planes used in this study can be found in [14].
Previously it was shown that $PG(2,16)$ contains exactly *two* unitals. Unitals in the rest of the planes of order 16 have not been completely classified, yet.
Known unitals in the known planes of order 16 were found by Stoichev and Tonchev (*thirty-eight* unitals) [15], Krčadinac and Smoljak (*three* unitals) [16], and Stoichev and Gezek (*one hundred and fifteen* unitals) [12].
A pedal set of a unital U in a plane of order 16 comes from 5 points of U. The following *five* configurations (denoted by their line types) are possible for pedal sets in these planes: Either all 5 points are on a line (5^1), or 4 points are on a line and *four* 2-lines ($4^1,2^4$), or *two* 3-lines and *four* 2-lines ($3^2,2^4$), or 3 points are on a line and *seven* 2-lines ($3^1,2^7$), or *ten* 2-lines (2^{10}). Possible geometries of these configurations could be found in [16, Figure 2].
Using the computational algebra system MAGMA [17], pedal sets of all known unitals in the known projective planes of order 16 have been calculated. The algorithm used in our computations contains the following steps:
Step 1: Define the set of *lines (L)*, *points (P)*, *unitals (U)* of the Plane π, and *line types (LT)*
Step 2: For each unital u in U do
Step 3: For each point $p \in P\backslash u$, find its *tangents (T)*
Step 4: For every tangent $t \in T$ find $t \cap u$ and save them in a set *ps* // *ps* is the set of pedal sets of the point p
Step 5: Save the pedal sets *ps* in an indexed set *PS*
Step 6: For each pedal set *ps* in *PS*, check which line type in *LT* it possesses
Step 7: Print the number of each possible line type
The number of known unitals in the known planes of order 16 is 156. Specific point sets of the known unitals used in this study can be found in [12]. Pedal sets of the *forty-two* of the known unitals are studied in [16]. We list the details of the pedal sets of the remaining unitals in Table 3.1, where Column 1 states the name of the plane, Column 2 provides the unital no's, and the last column gives the numbers of pedal sets for each type. All except 38 of unitals in these planes have the same pedal sets counts with their duals. Details of the pedal sets of dual unitals having different pedal set counts with their duals are listed in Table 3.2.
Table 3.1 shows that all unitals except unital 4 of BBH1 plane, unital 18 of BBH2 plane, unital 11 of BBS4 plane, all known unitals in DEMP plane, units 4, 6, 7, and 8 of MATH plane, unital 5 of JOWK plane, unitals 3, 4, 7 and 8 of SEMI2 plane and unitals 2, 3, and 7 of SEMI4 plane have at least 16 special points.
Previously, there were only *two* unitals in BBH1 plane having a special tangent. Our computations show that unitals 14 and 16 of BBH1 plane also possess a special tangent. All of the unitals having a special tangent in BBH1 plane has special points not lying on a special tangent: Unital 1 of BBH1 plane has *sixteen* special points outside of a special tangent, which are divided into *four* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 2 (and 14) of BBH1 plane has *fifty-two* special points outside of a special tangent. None of these points lies on a secant through the intersection point of the special tangent with the unital. Unital 16 of BBH1 plane has *eight* special points outside of a special tangent, which are divided into *two* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. *Eight* of the unitals of BBH1 plane contains exactly 16 special points, but none of these points lie on a special tangent.
BBH2 plane previously was known to contain only *one* unital having a special tangent. Our computations show that there are *six* more unitals in BBH2 plane having exactly one special tangent, all of which have special points not lying on a special tangent: Unitals 19, 20, 22, and 23 of BBH2 plane has *eight* special points outside of a special tangent, which are divided into *two* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 21 of BBH2 plane has *sixteen* special points outside of a special tangent, which are divided into *two* distinct sets of size 8 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 26 of BBH2 plane has *twenty-four* special points outside of a special tangent, which are divided into *six* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Table 3.1 shows that *seven* of the unitals in BBH2 plane have exactly 16 special points, but none of these points lie on a special tangent.
None of the known unitals in BBS4, DEMP, and DSFP planes have a special tangent, but *six* unitals in BBS4 plane have exactly 16 special points, but none of these points lie on a special tangent.
Details of the pedal sets of the known unitals in HALL plane can be found in [16]. Only unitals 4 and 6 of HALL plane contains special points not lying on a special tangent: Unital 4 of HALL plane has *sixteen* special points outside of a special tangent, which are divided into *four* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 6 of HALL plane has *fifty-two* special points outside of a special tangent, which are divided into *ten* distinct sets of size 4 and *one* set of size 12 such that each set lies on a secant through the intersection point of the special tangent with the unital.
All known unitals in MATH plane having exactly 16 special points, as well as unitals 5 and 13, have exactly one special tangent. Table 3.1 shows that unitals having more than 16 special points in MATH plane have special points not lying on a special tangent: Unital 5 of MATH plane has *eight* special points outside of a special tangent, which are divided into *two* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 13 of MATH plane has *sixty-four* special points outside of a special tangent, which are divided into *sixteen* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital.
JOHN plane contains three unitals having a special tangent, two of which have special points not lying on a special tangent: Unital 2 of JOHN plane has *sixteen* special points outside of a special tangent, which are divided into *four* distinct sets of size

4 such that each set lies on a secant through the intersection point of the special tangent with the unital. Unital 26 of JOHN plane has *eight* special points outside of a special tangent, which are divided into *two* distinct sets of size 4 such that each set lies on a secant through the intersection point of the special tangent with the unital.
The number of known unitals in SEMI2 plane is 21, all except four have exactly *sixteen* special points and a special tangent. None of the unitals in SEMI2 plane having at least 16 special points have special points outside of a special tangent.
SEMI4 plane previously was known to have exactly two unitals having a special tangent. Our computations show that many of the known unitals in SEMI4 plane possess a special tangent. Two of these unitals have special points not lying on a special tangent: Unital 4 of SEMI4 plane has *four* special points outside of a special tangent, which lies on a secant through the intersection point of the special tangent with the unital.
In [18], it was shown that special points and special tangents of a unital U give rise to parallel classes and resolutions of the unital design associated with U, respectively. Even though, all parallel classes and resolutions of the unital designs associated with a unital in planes of order 9 come from special points and special tangents, respectively [16], this is not true in general. The parallel classes of the designs associated with the following unitals in planes of order 16 come from special points: Unitals 1 and 16 of BBH1 plane, units 6, 21, and dual unitals 7, 20, and 26 of BBH2 plane, unital 11 of BBS4 plane, unital 3 of DEMP plane, unital 5 of HALL plane, all unitals in LMRH plane, all unitals except unitals 5 and 9 of MATH plane, unitals 2 and 29 and dual unital 26 of JOHN plane, unitals 5 and 7 of JOWK plane, all unitals except unitals 2 and 10 of SEMI2 plane, all unitals except unitals 3 and 4 of SEMI4 plane, and unital 2 of PG(2,16). The number of parallel classes of the designs associated with the rest of the known unitals in planes of order 16 is grater then the number of special points of unitals.

3. Conclusion

Previously, no unitals without special points were known to exist (a question asked by the authors in [16]), but the data given in Table 3.1 shows that unitals $6, 6^{\perp}, 7$, and 8 of MATH plane and unital 2 of SEMI4 plane do not have any special points (unital 2 of SEMI4 plane in [16] is the unital 12 in [12]).
All known unitals in projective planes of order $q^2 \in \{9, 16\}$ having at least one special tangent have the property that the number of special points is a multiple of q. Does this property hold in general?
Unitals 2 and 14 of BBH1 plane are the first (and only) examples of unitals having the following property: none of the special points outside of a special tangent lies on a secant through the intersection point of the special tangent with the unital. Why do these unitals act differently?
None of the unitals given in Table 3.2 have a special tangent. This shows that if a unital U in a plane of order 16 has a special tangent, then U and $U^{\perp}$ have the same pedal set counts. Unitals in planes of order 9 having at least one special tangent also possess this property [16]. Are there unitals not having this property?
It was observed in [16] that the number of pedal sets having line type $(q+1)$ always seems to agree for a unital and its dual unital. We notice that not only the number of line type $(q+1)$, but also the number of line type $(q, 2^q)$ seems to agree for a unital and its dual unital. Can we prove that this property holds in general?
We end this paper with the following conjecture:

Conjecture 1. *Let U be a unital embedded in a projective plane of order q^2, and $n_i(U)$ be the number of pedal sets of U having line type i. Then,*

$$n_i(U) = n_i(U^{\perp})$$

for $i \in \{(q+1), (q, 2^q)\}$. Furthermore, if U has a special tangent, then

$$n_i(U) = n_i(U^{\perp})$$

for any i.

Plane	Unital No.	Pedal set				
		(5)	$(4,2^4)$	$(3^2,2^4)$	$(3,2^7)$	(2^{10})
BBH1	4	11	0	24	84	89
	5	16	12	4	104	72
	6	16	12	4	104	72
	7	16	0	16	104	72
	8	28	24	12	96	48
	9	28	12	36	52	80
	10	28	24	12	96	48
	11	16	4	12	88	88
	12	16	4	12	88	88
	13	16	8	12	104	68
	14	68	0	0	104	36
	15	16	8	12	104	68
	16	24	16	16	96	56
BBH2	7	11	8	32	96	61
	8	16	24	4	92	72
	9	16	16	0	124	52
	10	28	24	0	108	48
	11	16	12	20	92	68
	12	16	4	28	80	80
	13	16	8	12	104	68
	14	20	16	12	96	64
	15	24	0	32	76	76
	16	24	4	20	92	68
	17	16	0	20	104	68
	18	8	0	30	120	50
	19	24	24	8	136	16
	20	24	56	8	88	32
	21	32	8	0	96	72
	22	24	24	0	88	72
	23	24	16	0	112	56
	24	16	24	12	112	44
	25	32	16	12	88	60
	26	40	16	0	112	40
BBS4	2	16	16	4	76	96
	3	16	24	0	76	92
	4	20	12	4	88	84
	5	16	16	8	92	76
	6	20	8	4	96	80

Table 3.1: Pedal sets of unitals in planes of order 16.

Plane	Unital No.	Pedal set				
		(5)	$(4,2^4)$	$(3^2,2^4)$	$(3,2^7)$	(2^{10})
BBS4	7	24	16	24	76	68
	8	24	8	12	100	64
	9	20	16	36	92	44
	10	16	24	20	52	96
	11	4	0	0	150	54
	12	16	0	0	156	36
	13	16	24	48	60	60
DEMP	3	4	0	24	24	156
	4	4	12	12	144	36
MATH	5	24	8	0	56	120
	6	0	0	32	48	128
	7	0	0	32	80	96
	8	0	0	16	32	160
	9	12	12	44	64	76
	10	16	0	0	192	0
	11	16	0	0	0	192
	12	16	0	0	0	192
	13	80	0	0	128	0
	14	16	0	0	64	128
	15	16	0	0	64	128
	16	16	0	0	64	128
JOHN	6	16	20	0	68	104
	7	16	16	0	80	96
	8	16	12	4	80	96
	9	16	8	0	116	68
	10	20	12	0	96	80
	11	20	16	0	92	80
	12	24	0	16	100	68
	13	20	0	16	64	108
	14	16	20	20	112	40
	15	20	12	12	68	96
	16	20	16	12	112	48
	17	16	24	20	76	72
	18	24	4	20	88	72
	19	16	44	12	84	52
	20	16	32	16	100	44
	21	20	0	16	88	84
	22	24	12	20	60	92

Table 3.1: (Continued)

Plane	Unital No.	Pedal set				
		(5)	$(4,2^4)$	$(3^2,2^4)$	$(3,2^7)$	(2^{10})
JOHN	23	20	12	12	80	84
	24	24	8	12	64	100
	25	20	12	20	52	104
	26	24	0	0	128	56
	27	20	0	12	112	64
	28	24	0	12	88	84
	29	16	0	16	48	128
JOWK	5	4	0	12	84	108
	6	20	0	12	96	80
	7	16	0	16	128	48
SEMI2	4	4	0	12	48	144
	5	16	0	32	96	64
	6	16	0	0	160	32
	7	4	0	12	144	48
	8	4	0	60	48	96
	9	16	0	32	128	32
	10	16	0	0	96	96
	11	16	0	0	64	128
	12	16	0	64	64	64
	13	16	0	0	192	0
	14	16	0	0	0	192
	15	16	0	0	0	192
	16	16	64	64	64	0
	17	16	0	0	64	128
	18	16	0	0	0	192
	19	16	0	0	0	192
	20	16	0	0	192	0
	21	16	0	0	192	0
SEMI4	2	0	16	16	64	112
	3	4	4	12	100	88
	4	20	16	0	96	76
	5	16	24	0	72	96
	6	16	24	0	72	96
	7	4	0	12	144	48
	8	16	0	0	128	64
	9	16	64	0	64	64
	10	16	0	0	128	64
	11	16	64	0	64	64

Table 3.1: (Continued)

Plane	Unital No.	Pedal set				
		(5)	$(4,2^4)$	$(3^2,2^4)$	$(3,2^7)$	(2^{10})
BBH2	7	11	8	24	112	53
	8	16	24	0	100	68
	11	16	12	12	108	60
	12	16	4	20	96	72
	14	20	16	20	80	72
	18	8	0	35	110	55
	23	24	16	16	80	72
BBS4	2	16	16	0	84	92
	4	20	12	0	96	80
	5	16	16	4	100	72
	6	20	8	0	104	76
	9	20	16	32	100	40
	10	16	24	28	36	104
	13	16	24	24	108	36
DEMP	3	4	0	12	48	144
MATH	6	0	0	16	80	112
JOHN	6	16	20	4	60	108
	8	16	12	0	88	92
	9	16	8	8	100	76
	10	20	12	4	88	84
	11	20	16	4	84	84
	17	16	24	24	68	76
	19	16	44	20	68	60
	21	20	0	24	72	92
	24	24	8	20	48	108
	25	20	12	12	68	96
	26	24	0	8	112	64
JOWK	6	20	0	28	64	96

Table 3.2: Pedal sets of the dual unitals in planes of order 16.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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