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# Nonsingular cubic surfaces over $\mathbb{F}_{2^k}$

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**Abstract:** We perform an opportunistic search for cubic surfaces over small fields of characteristic two. The starting point of our work is a list of surfaces complied by Dickson over the field with two elements. We consider the nonsingular ones arising in Dickson' s work for the fields of larger orders of characteristic two. We investigate the properties such as the number of lines, singularities and automorphism groups. The problem of determining the possible numbers of lines of a nonsingular cubic surface over the fields of $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{F}_q$ where q odd, $\mathbb{F}_2$ was considered by Cayley and Salmon, Schläfli, Segre, Rosati and Dickson, respectively. Our work contributes this problem over the larger fields of even characteristic. Besides that we investigate the structure of nonsingular surfaces with 15 and 9 lines. This work is a contribution to the study of nonsingular cubic surfaces with less than 27 lines.

**Key words:** Geometry, finite field, cubic surface, number of lines, Dickson surfaces

## 1. Overview

In [11], Dickson classifies the nonsingular cubic surfaces over the field $\mathbb{F}_2$ and gives some examples of singular surfaces. Cooley in [9] determines the total number of cubic surfaces over $\mathbb{F}_2$, singular or not, by an application of Burnside's Lemma. In this paper, we classify all cubic surfaces over the field $\mathbb{F}_2$ by constructing a complete set of representatives of the isomorphism types, thereby confirming the results of both Dickson and Cooley. Besides, we investigate the 36 nonsingular cubic surfaces arising in Dickson' s work over small extension fields. We extend Dickson's study by determining the automorphism groups of each surface. We distinguish between the stabiliser of the set of points on a surface and the stabiliser of the equation. We investigate the relationship between the two groups.

By an old result of Cayley and Salmon [7], a nonsingular cubic surface over an algebraically closed field has exactly 27 lines. Because $\mathbb{R}, \mathbb{Q}$ and finite fields are not algebraically closed, the number of lines on a nonsingular cubic surface varies. The number of lines of a nonsingular cubic surface must be one of the following over real numbers $3, 7, 15$ or $27$, by Schläfli [20], over rational numbers $0, 1, 2, 3, 5, 7, 9, 15$ or $27$, by Segre [22], over the finite field of odd characteristic $0, 1, 2, 3, 5, 7, 9, 15$ or $27$, by Rosati [19], over the finite field of two elements $0, 1, 2, 3, 5, 9$ or $15$, by Dickson [11]. Segre in [22] showed that the number of lines on a nonsingular cubic surface over any field must be one of 0, 1, 2, 3, 5, 7, 9, 15 and 27. We observed that the number of lines of smooth cubic surface over fields of characteristic two $\mathbb{F}_{2^k}$ can be one of the following: $0, 1, 2, 3, 5, 7, 9, 15, 27$. We also observed that there are some nonsingular surfaces with 4 lines, for instance the surface given by the equation

$$x_2^3 + x_3^3 + x_0^2 x_2 + x_0^2 x_3 + x_1^2 x_2 + x_0 x_1 x_2 = 0$$

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2010 *AMS Mathematics Subject Classification:* 05B25, 05E18, 14E05, 14G27 23584 over $\mathbb{F}_4$. However, two singular points of this surface appear over $\mathbb{F}_{16}$. We do not take this kind of cases into the account since some singularities arise over some other extension field of $\mathbb{F}_2$. Therefore, we can never complete this surface to have 27 lines over algebraic closure. So, we mean by saying nonsingular is in fact absolutely nonsingular.

Several of the surfaces arising in Dickson's study have been generalised to infinite families of cubic surfaces with 27 lines. One family, due to Hirschfeld [15], exists whenever the field contains $\mathbb{F}_4$ as a subfield. Recently, in joint work with Betten, further families of cubic surfaces with 27 lines in characteristic two have been noted[*],[†]. Several of these families reduce to cases in Dickson's list. For this reason, it seems justified to study the surfaces in Dickson's list over small extension fields of $\mathbb{F}_2$.

This paper is based on experimential data computed using a computer algebra system. Our main tool is Orbiter[‡] which can be used to both classify and investigate cubic surfaces over small finite fields. In order to classify cubic surfaces, tools from Computational Group Theory ([6], [16], [23]) will be used. With these tools at hand, the work of Dickson can be verified by computer in a matter of minutes.

## 2. Introduction

A cubic surface $\mathcal{F}$ over the field $K$ is the zero set of a homogeneous cubic equation in 4 variables over $K$. For instance,

$$\mathcal{F} = V\big(x_0^3 + x_1^3 + x_2^3 + x_3^3 - (x_0 + x_1 + x_2 + x_3)^3\big)$$

is known as the Clebsch surface. Other examples are the Fermat surface, Hilbert Cohn Vossen surface. In this paper, we consider cubic surfaces over fields of characteristic two. A singular point of $\mathcal{F}$ is a point of $\mathcal{F}$ where all partial derivatives vanish. For us, a surface is nonsingular if it does not have any singular point over the field $K$ and over any extension field of $K$.

An important tool to study cubic surfaces is the birational structure. This means that there are rational maps between the surface and a projective plane. The main property of these maps is that there is an exceptional locus on both the surface and the plane [12].

Let $\mathbb{F}_q$ denote the finite field of order $q = p^k$ where $p$ is a prime. Let $\overline{\mathbb{F}_p}$ denote the algebraic closure of $\mathbb{F}_p$, i.e $\mathbb{F}_p$ adjoint all roots of polynomials over $\mathbb{F}_p$. $\overline{\mathbb{F}_p}$ is an algebraically closed field of characteristic $p$. Every $\mathbb{F}_p$ has a unique $\overline{\mathbb{F}_p}$. $\overline{\mathbb{F}_p}$ contains every $\mathbb{F}_{p^e}$ for all $e \geq 1$.

$\mathrm{PG}(n,q)$ is the $n$-dimensional projective space over $\mathbb{F}_q$. The projective plane $\mathrm{PG}(2,q)$ over $\mathbb{F}_q$ contains $q^2+q+1$ points and $q^2+q+1$ lines. The space $\mathrm{PG}(3,q)$ contains $q^3+q^2+q+1$ points and equally many planes, as well as $(q^2+q+1)(q^2+1)$ lines. In $\mathrm{PG}(3,2)$, there are 15 points and 35 lines.

The automorphism group of $\mathrm{PG}(n,q)$ is the group of bijective mappings which preserve collinearity (also called collineations). The Fundamental theorem of projective geometry says that

$$Aut(\mathrm{PG}(n,q)) = P\Gamma L(n+1,q), \text{ for } n \geq 2.$$

The collineation group contains as subgroup the group $\mathrm{PGL}(n+1,q)$, the group of projectivities of $\mathrm{PG}(n,q)$.

[*]Betten, A. and Karaoglu, F., Cubic Surfaces with 13 Eckardt Points, Submitted.

[†] Betten, A. and Karaoglu, F., The Eckardt Point Configurations of Cubic Surfaces Revisited, Submitted

[‡]Betten(2021). Orbiter – A program to classify discrete objects [online]. Website https://github.com/abetten/orbiter [accessed 02 February 2021].

The order of the group of projectivities of $\mathrm{PG}(3,q)$ is

$$|\mathrm{PGL}(4,q)| = \frac{(q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3)}{q-1}. \tag{2.1}$$

Let $\mathcal{F}$ be a nonsingular cubic surface defined over the finite field $\mathbb{F}_q$. It is well known that the number of points on $\mathcal{F}$ satisfies $q^2+tq+1$ where $-2 \leq t \leq 7$ and $t=6$ is impossible [18]. The case of $t=7$ is equivalent to the cubic surface having exactly 27 lines. Hirschfeld in [13] shows that this is impossible for $q=2,3$ and 5. This question has been reconsidered recently in [24].

Das in [10] gives the formulas for the number of nonsingular cubic surfaces with $q^2+tq+1$ points for each possible $t$. When $t=7$, this formula is given in Elkies' study[§] and Karaoglu and Betten's paper [17] with different approaches. These numbers can be used to verify the classification results. Let $C_q^t$ be the number of nonsingular cubic surfaces $\mathcal{F}$ with $q^2+tq+1$ points and $g$ be the order of $\mathrm{PGL}(4,q)$. Let $G(\mathcal{F})$ be the group of projectivities of such cubic surface $\mathcal{F}$. By Orbit–Stabiliser theorem

$$C_q^t := \sum_{\text{iso type } \mathcal{F}} \frac{g}{|G(\mathcal{F})|}. \tag{2.2}$$

The automorphism group of a surface $\mathcal{F}$ is the stabiliser of $\mathcal{F}$. We distinguish between the collineation stabiliser $\Gamma(\mathcal{F})$ and the projective stabiliser $G(\mathcal{F})$. Furthermore, we distinguish between the stabiliser of the equation of $\mathcal{F}$ and the stabiliser of the associated set of points on $\mathcal{F}$. These two groups can differ as we will see in our work.

Next, we will review some structural properties of cubic surfaces.

There are at least two ways to define lines. One is the combinatorial definition where a line is considered as a subsets of points on the surface. Another one is algebraic definition where the equation of the surface vanishes when restricted to the line. Dickson calls such a line real line.

While a nonsingular cubic surface over an algebraically closed field has precisely 27 lines, a nonsingular cubic surface over a finite field can have fewer lines. Rosati in [19] shows that this number must be one of $27, 15, 9, 7, 5, 3, 2, 1, 0$ when $q$ is odd. Dickson in [11] show that a nonsingular cubic surface over $\mathbb{F}_2$ can have $15, 9, 5, 3, 2, 1,$ or 0 lines. Our work will show that 7 is possible for $q=2^k$ as well.

Schläfli [20] introduces a very useful notation to label the 27 lines of a cubic surface, should the surface contain them. These labels extend to Eckardt points, tritangent planes, trihedral pairs and triads etc. Schläfli further observes the special rule played by certain sets of twelve lines called double-sixes. A double-six in $\mathrm{PG}(3,K)$ is the set of 12 lines

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix} \tag{2.3}$$

such that $a_i$ intersects $b_j$ if and only if $i \neq j$. In addition, the $a_i$ are pairwise skew and the $b_i$ are pairwise skew also. The main property of a double-six is that it determines a unique cubic surface with 27 lines. The remaining 15 lines are denoted as $c_{ij}$, where $c_{ij}$ is the line of intersection of $[a_i, b_j]$ and $[a_j, b_i]$, with $[a_i, b_j]$ being the plane spanned by $a_i$ and $b_j$. A cubic surface with 27 lines gives rise to exactly 36 double-sixes, neglecting permutations of rows and columns within the scheme (2.3) [14].

[§]Elkies, N. D., Linear codes and algebraic geometry in higher dimensions. (Preprint, 2006).

Eckardt point, double point and single point of the cubic surface is the point which it respectively meets three, two and one lines of the cubic surface. There are some invariants to distinguish the cubic surfaces such as their singularity type; the number of lines, points, Eckardt points, double points, single points on the cubic surface and the number of points not on any line of the surface.

There is a relation between cubic surfaces and certain configurations of points in the plane [8]. The relation is induced by the birational structure of the surface (see [2]). In [17], this map is refered to as the Clebsch map and the following notation is established. The Clebsch map $\Phi_{\pi,\ell_1,\ell_2}$ from a surface $\mathcal{F}$ to a plane $\pi$ is defined by picking one tritangent plane $\pi$ and two skew lines $\ell_1$, $\ell_2$ on the surface not contained in $\pi$. The main property of this map is that it sends a half double-six to a 6-arc not on a conic. Here, a half double-six refers to one of the two rows of 6 lines in a double-six. The map $\Phi_{\pi,\ell_1,\ell_2}$ is one to one and onto, outside of the half double-six.

There are many Clebsch maps. It is possible to choose one which sends all points of a given half double-six to 6 points $P_1,\dots,P_6$ in the plane $\pi$. Considering this map and the fact that there are $q+1$ point on a line and there are $q^2+q+1$ points on the plane, the following count give us the number of points on the cubic surface: $q^2+q+1-6+6(q+1)=q^2+7q+1$. So, a cubic surface with 27 lines has $q^2+7q+1$ points.

Let $S$ be the set of all objects of type $X$ and $G$ be the group which acts on the set $S$. Classification of $G$ on $S$ means to partition $S$ into classes, called orbits. Two elements belong to the same orbit if and only if one can be moved to the other under an element $g \in G$. In this paper, we are interested in the classification of cubic surfaces over a finite field $\mathbb{F}_q$ under the projective group $\mathrm{PGL}(4,q)$. While [4] is concerned with cubic surfaces with 27 lines, the present work is directed at studying all cubic surfaces over a given field $\mathbb{F}_q$, singular or not.

There are many different ways to do isomorphism testing for the cubic surfaces. For cubic surfaces with 27 lines, the algorithms described in [4] are available. For cubic surfaces with fewer than 27 lines, the situation is less satisfying. Graph theoretic techniques similar to those of [1] can be used for very small finite fields only. To make matters worse, these methods compute the set theoretic stabiliser of a surface, which may or may not be equal to the stabiliser of the surface equation. Finally, the method of Schreier trees from Computational Group Theory ([6],[16],[23]) can be used to compute orbits and stabilisers. However, this method is even less efficient. This algorithm is implemented in many Computer Algebra Systems such as Magma [5] Orbiter[§] and GAP[¶]. In order to compute the Schreier tree, one needs a generating set of the group. For the projective linear and semilinear groups, such a generating set has been described in [3, Chapter 9]. In fact, this generating set is a strong generating set. This means that there is a chain of subgroups, each of which is the stabiliser of a point in the permutation domain in the previous group. The sequence of points arising in this chain of subgroups is called a base for the permutation group. A generating set is called strong if it contains sufficiently many elements to generate each group in the chain.

A family is a class of objects with the same construction method. The Hirschfeld family exists over $\mathbb{F}_{4^k}$ [15]. The equation of the surface is

$$V(x_0^2x_1+x_0x_1^2+x_2^2x_3+x_2x_3^2).$$

It has 45 Eckardt points and its automorphism group is associated with the Weyl Group of type $E6$. All cubic surfaces with 27 lines and 13 Eckardt points have been described in[†]. One example in [†] is the Tekirdağ 1

[¶]The GAP Group(2017). GAP – Groups, Algorithms, and Programming, Version 4.8.7 [online]. Website http://www.gap-system.org family, which exists over $\mathbb{F}_{8^k}$. The equation of this surface is

$$V(\gamma^2\left[X_0^2X_3 + X_0X_3^2\right] + \gamma^6\left[X_1^2X_2 + X_1X_2^2\right] + X_1X_2X_3),$$

where $\gamma$ is the primitive element of $\mathbb{F}_8$. It has 13 Eckardt points and the projective stabiliser has order 192.

## 3. Projective classification of cubic surfaces modulo 2

For our purposes, it is important to revisit the classification of cubic surfaces over $\mathbb{F}_2$. We used Orbiter[¶] to recreate the classification and to match it up with Dickson's work. This will produce a table of the nonsingular cubic surfaces over $\mathbb{F}_2$. In Section 4, we will consider these surfaces over extension fields of $\mathbb{F}_2$.

**Theorem 3.1** *[11] There are* 36 *nonsingular cubic surfaces over* $\mathbb{F}_2$.

Summary of Dickson' s proof [11]:

There are 15 points in $\mathrm{PG}(3,2)$.

$$\begin{array}{ccccc} 1=(1,0,0,0) & 2=(0,1,0,0) & 3=(0,0,1,0) & 4=(0,0,0,1) & 5=(1,1,0,0) \\ 6=(1,0,1,0) & 7=(1,0,0,1) & 8=(0,1,1,0) & 9=(0,1,0,1) & 10=(0,0,1,1) \\ 11=(1,1,1,0) & 12=(1,1,0,1) & 13=(1,0,1,1) & 14=(0,1,1,1) & 15=(1,1,1,1) \end{array}$$

Let $\mathcal{F}$ be the cubic surface over $\mathbb{F}_2$. $\mathcal{F}$ can be identified with a set of $N$ points in $\mathrm{PG}(3,2)$ for some $N$. Dickson's approach first classifies all $N$ subsets for all possible $N$ (which must be odd for $q=2$) of $\mathrm{PG}(3,2)$ under the group $\mathrm{PGL}(4,2)$. After that, he continues to classify the nonsingular cubic surfaces associated with each of the sets that were chosen as orbit representatives in the first step. His isomorphism testing is based on case by case arguments.

**Theorem 3.2** *[9] There are* 141 *cubic surfaces over* $\mathbb{F}_2$*, singular or not.*

**Remark 3.3** *Cooley's proof is based on an application of Burnside's Lemma from enumerative combinatorics. This does not give the equations of the surfaces.*

**Theorem 3.4** *The* 141 *cubic surfaces over* $\mathbb{F}_2$ *are listed in Table 1 together with their stabiliser group orders, their number of lines, points, singular points, Eckardt points, double points. The distribution of number of points on the surfaces is*

$$15^2,\ 13^5,\ 11^{19},\ 9^{39},\ 7^{45},\ 5^{22},\ 3^7,\ 1^2.$$

*In Table 1, the surfaces are denoted by* $F_i$ *where* $i \in \{1,\dots,141\}$. *The nonsingular surfaces are* $F_1,\dots,F_{36}$. *The equations of* $F_1,\dots,F_{36}$ *are listed in Table 2. The equations of the remaining surfaces can be found in the data set associated with this paper* [‖]*, as well as some of important invariants of the surfaces.*

**Proof** Every cubic surface can be identified with its coefficient vector, which is an element in $\mathrm{PG}(19,q)$. In the problem of classification of cubic surfaces over $\mathbb{F}_2$, the group is $\mathrm{PGL}(4,2)$ of order 20160 acting on the

[‖] Karaoglu (2021). Atlas of Dickson Surfaces. [online] Website https://ftmakroglu05.github.io/DicksonSurfaces/ [accessed 15 January 2021].

**Table 1**. All cubic surfaces over $\mathbb{F}_2$.

| $\mathcal{F}$ | #S | #$\ell$ | #$P$ | #E | #D | $\lvert\mathrm{Aut}(\mathcal{F})\rvert$ | $\mathcal{F}$ | #S | #$\ell$ | #$P$ | #E | #D | $\lvert\mathrm{Aut}(\mathcal{F})\rvert$ | $\mathcal{F}$ | #S | #$\ell$ | #$P$ | #E | #D | $\lvert\mathrm{Aut}(\mathcal{F})\rvert$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_1$ | 0 | 15 | 15 | 15 | 0 | 720 | $F_{48}$ | 1 | 10 | 13 | 6 | 0 | 48 | $F_{95}$ | 1 | 0 | 7 | 0 | 0 | 3 |
| $F_2$ | 0 | 9 | 13 | 4 | 6 | 12 | $F_{49}$ | 1 | 10 | 13 | 2 | 9 | 8 | $F_{96}$ | 1 | 0 | 5 | 0 | 0 | 3 |
| $F_3$ | 0 | 5 | 11 | 2 | 0 | 16 | $F_{50}$ | 1 | 7 | 11 | 7 | 0 | 24 | $F_{97}$ | 1 | 0 | 3 | 0 | 0 | 2 |
| $F_4$ | 0 | 5 | 11 | 1 | 3 | 2 | $F_{51}$ | 1 | 6 | 11 | 2 | 3 | 6 | $F_{98}$ | 1 | 0 | 7 | 0 | 0 | 2 |
| $F_5$ | 0 | 3 | 9 | 1 | 0 | 12 | $F_{52}$ | 1 | 6 | 11 | 1 | 6 | 3 | $F_{99}$ | 1 | 0 | 5 | 0 | 0 | 6 |
| $F_6$ | 0 | 3 | 9 | 1 | 0 | 4 | $F_{53}$ | 1 | 6 | 11 | 1 | 6 | 6 | $F_{100}$ | 1 | 0 | 5 | 0 | 0 | 3 |
| $F_7$ | 0 | 3 | 9 | 0 | 3 | 2 | $F_{54}$ | 1 | 6 | 11 | 0 | 4 | 2 | $F_{101}$ | 1 | 0 | 5 | 0 | 0 | 1 |
| $F_8$ | 0 | 3 | 9 | 0 | 0 | 18 | $F_{55}$ | 1 | 5 | 11 | 0 | 0 | 32 | $F_{102}$ | 2 | 8 | 11 | 3 | 7 | 4 |
| $F_9$ | 0 | 3 | 7 | 1 | 0 | 48 | $F_{56}$ | 1 | 4 | 9 | 0 | 0 | 16 | $F_{103}$ | 2 | 7 | 11 | 1 | 5 | 2 |
| $F_{10}$ | 0 | 2 | 9 | 0 | 0 | 1 | $F_{57}$ | 1 | 4 | 9 | 0 | 4 | 2 | $F_{104}$ | 2 | 5 | 9 | 1 | 4 | 1 |
| $F_{11}$ | 0 | 2 | 9 | 0 | 0 | 1 | $F_{58}$ | 1 | 4 | 9 | 1 | 1 | 4 | $F_{105}$ | 2 | 4 | 9 | 0 | 4 | 2 |
| $F_{12}$ | 0 | 1 | 9 | 0 | 0 | 6 | $F_{59}$ | 1 | 4 | 9 | 1 | 1 | 4 | $F_{106}$ | 2 | 4 | 9 | 1 | 2 | 1 |
| $F_{13}$ | 0 | 1 | 9 | 0 | 0 | 1 | $F_{60}$ | 1 | 4 | 9 | 0 | 4 | 2 | $F_{107}$ | 2 | 4 | 9 | 0 | 4 | 2 |
| $F_{14}$ | 0 | 1 | 7 | 0 | 0 | 4 | $F_{61}$ | 1 | 3 | 7 | 1 | 0 | 48 | $F_{108}$ | 2 | 3 | 7 | 0 | 3 | 2 |
| $F_{15}$ | 0 | 1 | 7 | 0 | 0 | 2 | $F_{62}$ | 1 | 3 | 7 | 1 | 0 | 24 | $F_{109}$ | 2 | 3 | 7 | 0 | 2 | 4 |
| $F_{16}$ | 0 | 1 | 7 | 0 | 0 | 2 | $F_{63}$ | 1 | 3 | 9 | 0 | 2 | 2 | $F_{110}$ | 2 | 3 | 7 | 0 | 2 | 2 |
| $F_{17}$ | 0 | 1 | 7 | 0 | 0 | 1 | $F_{64}$ | 1 | 3 | 7 | 0 | 3 | 2 | $F_{111}$ | 2 | 2 | 7 | 0 | 1 | 1 |
| $F_{18}$ | 0 | 1 | 7 | 0 | 0 | 1 | $F_{65}$ | 1 | 3 | 7 | 0 | 3 | 6 | $F_{112}$ | 2 | 2 | 7 | 0 | 1 | 2 |
| $F_{19}$ | 0 | 1 | 7 | 0 | 0 | 2 | $F_{66}$ | 1 | 3 | 9 | 0 | 3 | 2 | $F_{113}$ | 2 | 2 | 7 | 0 | 1 | 2 |
| $F_{20}$ | 0 | 1 | 5 | 0 | 0 | 2 | $F_{67}$ | 1 | 3 | 9 | 0 | 2 | 2 | $F_{114}$ | 2 | 2 | 7 | 0 | 1 | 4 |
| $F_{21}$ | 0 | 1 | 5 | 0 | 0 | 4 | $F_{68}$ | 1 | 3 | 9 | 1 | 0 | 4 | $F_{115}$ | 2 | 1 | 5 | 0 | 0 | 2 |
| $F_{22}$ | 0 | 1 | 3 | 0 | 0 | 48 | $F_{69}$ | 1 | 3 | 9 | 0 | 2 | 2 | $F_{116}$ | 3 | 19 | 15 | 12 | 0 | 576 |
| $F_{23}$ | 0 | 0 | 9 | 0 | 0 | 24 | $F_{70}$ | 1 | 3 | 9 | 0 | 3 | 1 | $F_{117}$ | 3 | 13 | 13 | 4 | 6 | 12 |
| $F_{24}$ | 0 | 0 | 7 | 0 | 0 | 6 | $F_{71}$ | 1 | 2 | 7 | 0 | 1 | 2 | $F_{118}$ | 3 | 9 | 11 | 6 | 0 | 16 |
| $F_{25}$ | 0 | 0 | 7 | 0 | 0 | 2 | $F_{72}$ | 1 | 2 | 5 | 0 | 1 | 16 | $F_{119}$ | 3 | 8 | 9 | 6 | 0 | 48 |
| $F_{26}$ | 0 | 0 | 7 | 0 | 0 | 1 | $F_{73}$ | 1 | 2 | 7 | 0 | 1 | 2 | $F_{120}$ | 3 | 8 | 11 | 2 | 6 | 2 |
| $F_{27}$ | 0 | 0 | 7 | 0 | 0 | 1 | $F_{74}$ | 1 | 2 | 7 | 0 | 1 | 2 | $F_{121}$ | 3 | 7 | 7 | 7 | 0 | 192 |
| $F_{28}$ | 0 | 0 | 5 | 0 | 0 | 5 | $F_{75}$ | 1 | 2 | 5 | 0 | 1 | 8 | $F_{122}$ | 3 | 7 | 9 | 7 | 0 | 12 |
| $F_{29}$ | 0 | 0 | 5 | 0 | 0 | 1 | $F_{76}$ | 1 | 2 | 9 | 0 | 1 | 4 | $F_{123}$ | 3 | 6 | 9 | 0 | 9 | 6 |
| $F_{30}$ | 0 | 0 | 5 | 0 | 0 | 2 | $F_{77}$ | 1 | 2 | 7 | 0 | 1 | 2 | $F_{124}$ | 3 | 5 | 9 | 0 | 6 | 6 |
| $F_{31}$ | 0 | 0 | 5 | 0 | 0 | 2 | $F_{78}$ | 1 | 2 | 7 | 0 | 0 | 1 | $F_{125}$ | 3 | 5 | 9 | 2 | 2 | 2 |
| $F_{32}$ | 0 | 0 | 5 | 0 | 0 | 4 | $F_{79}$ | 1 | 2 | 9 | 0 | 1 | 2 | $F_{126}$ | 3 | 5 | 9 | 1 | 4 | 2 |
| $F_{33}$ | 0 | 0 | 5 | 0 | 0 | 1 | $F_{80}$ | 1 | 1 | 3 | 0 | 0 | 32 | $F_{127}$ | 3 | 5 | 9 | 1 | 4 | 2 |
| $F_{34}$ | 0 | 0 | 3 | 0 | 0 | 18 | $F_{81}$ | 1 | 1 | 7 | 0 | 0 | 4 | $F_{128}$ | 3 | 4 | 9 | 0 | 0 | 48 |
| $F_{35}$ | 0 | 0 | 3 | 0 | 0 | 2 | $F_{82}$ | 1 | 1 | 7 | 0 | 0 | 6 | $F_{129}$ | 3 | 4 | 9 | 0 | 0 | 24 |
| $F_{36}$ | 0 | 0 | 1 | 0 | 0 | 24 | $F_{83}$ | 1 | 1 | 5 | 0 | 0 | 1 | $F_{130}$ | 3 | 3 | 7 | 1 | 0 | 8 |
| $F_{37}$ | 0 | 5 | 11 | 1 | 2 | 8 | $F_{84}$ | 1 | 1 | 7 | 0 | 0 | 4 | $F_{131}$ | 3 | 3 | 7 | 0 | 3 | 6 |
| $F_{38}$ | 0 | 4 | 11 | 1 | 1 | 4 | $F_{85}$ | 1 | 1 | 5 | 0 | 0 | 2 | $F_{132}$ | 3 | 3 | 7 | 0 | 2 | 4 |
| $F_{39}$ | 0 | 4 | 11 | 0 | 4 | 2 | $F_{86}$ | 1 | 1 | 7 | 0 | 0 | 1 | $F_{133}$ | 3 | 2 | 5 | 0 | 1 | 16 |
| $F_{40}$ | 0 | 2 | 9 | 0 | 1 | 2 | $F_{87}$ | 1 | 1 | 7 | 0 | 0 | 1 | $F_{134}$ | 3 | 1 | 3 | 0 | 0 | 288 |
| $F_{41}$ | 0 | 2 | 9 | 0 | 1 | 2 | $F_{88}$ | 1 | 1 | 7 | 0 | 0 | 1 | $F_{135}$ | 4 | 10 | 11 | 5 | 0 | 6 |
| $F_{42}$ | 0 | 2 | 7 | 0 | 1 | 4 | $F_{89}$ | 1 | 1 | 5 | 0 | 0 | 6 | $F_{136}$ | 4 | 9 | 11 | 5 | 6 | 24 |
| $F_{43}$ | 0 | 1 | 9 | 0 | 0 | 6 | $F_{90}$ | 1 | 1 | 5 | 0 | 0 | 4 | $F_{137}$ | 5 | 11 | 11 | 3 | 4 | 8 |
| $F_{44}$ | 0 | 1 | 7 | 0 | 0 | 6 | $F_{91}$ | 1 | 1 | 7 | 0 | 0 | 1 | $F_{138}$ | 5 | 8 | 9 | 6 | 0 | 16 |
| $F_{45}$ | 0 | 1 | 7 | 0 | 0 | 4 | $F_{92}$ | 1 | 0 | 5 | 0 | 0 | 3 | $F_{139}$ | 7 | 18 | 13 | 6 | 0 | 48 |
| $F_{46}$ | 0 | 0 | 7 | 0 | 0 | 21 | $F_{93}$ | 1 | 0 | 5 | 0 | 0 | 2 | $F_{140}$ | 7 | 13 | 11 | 8 | 0 | 96 |
| $F_{47}$ | 0 | 0 | 3 | 0 | 0 | 3 | $F_{94}$ | 1 | 0 | 1 | 0 | 0 | 168 | $F_{141}$ | 7 | 7 | 7 | 7 | 0 | 1344 |

**Table 2**. Equations of nonsingular cubic surfaces over $\mathbb{F}_2$.

| | |
|---|---|
| $F_1$ | $X_0^2X_3 + X_1^2X_2 + X_1X_2^2 + X_0X_3^2 = 0$ |
| $F_2$ | $X_0^2X_3 + X_1^2X_2 + X_1X_2^2 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_3$ | $X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_2 + X_0^2X_3 + X_0X_1^2 = 0$ |
| $F_4$ | $X_0^2X_3 + X_1^2X_3 + X_1X_2^2 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_5$ | $X_0^3 + X_0^2X_3 + X_1^2X_2 + X_1X_2^2 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_6$ | $X_0^3 + X_1^3 + X_2^3 + X_0^2X_3 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_7$ | $X_0^3 + X_2^3 + X_3^3 + X_0^2X_3 + X_0X_1^2 + X_0X_1X_2 = 0$ |
| $F_8$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_3 + X_1^2X_2 = 0$ |
| $F_9$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$ |
| $F_{10}$ | $X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_3 + X_0X_1X_2 = 0$ |
| $F_{11}$ | $X_0^2X_2 + X_1^2X_3 + X_1X_2^2 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_{12}$ | $X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_3 + X_0X_1^2 + X_1^2X_2 = 0$ |
| $F_{13}$ | $X_0^3 + X_2^3 + X_0^2X_1 + X_1^2X_3 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_{14}$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_2 = 0$ |
| $F_{15}$ | $X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0X_1X_2 = 0$ |
| $F_{16}$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0X_1X_2 = 0$ |
| $F_{17}$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_3 + X_1^2X_2 + X_0X_1X_2 = 0$ |
| $F_{18}$ | $X_2^3 + X_0^2X_2 + X_1^2X_3 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_{19}$ | $X_0^3 + X_0^2X_3 + X_1^2X_3 + X_1X_2^2 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_{20}$ | $X_0^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_3 + X_0X_1^2 + X_0X_1X_2 = 0$ |
| $F_{21}$ | $X_1^3 + X_2^3 + X_0^2X_3 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_{22}$ | $X_0^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_2 + X_0^2X_3 + X_0X_1^2 = 0$ |
| $F_{23}$ | $X_1^3 + X_2^3 + X_0^2X_1 + X_0^2X_2 + X_0^2X_3 + X_1^2X_2 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_{24}$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 = 0$ |
| $F_{25}$ | $X_0^3 + X_2^3 + X_3^3 + X_0^2X_3 + X_1^2X_2 + X_0X_1X_2 = 0$ |
| $F_{26}$ | $X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_3 + X_1^2X_2 + X_0X_1X_2 = 0$ |
| $F_{27}$ | $X_0^3 + X_2^3 + X_0^2X_2 + X_1^2X_3 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_{28}$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_3 + X_1^2X_2 = 0$ |
| $F_{29}$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_3 + X_0X_1X_2 = 0$ |
| $F_{30}$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_2 + X_0^2X_3 + X_0X_1X_2 = 0$ |
| $F_{31}$ | $X_0^3 + X_1^3 + X_3^3 + X_0^2X_3 + X_1^2X_3 + X_0X_2^2 + X_0X_1X_2 = 0$ |
| $F_{32}$ | $X_1^3 + X_2^3 + X_0^2X_3 + X_1^2X_2 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_{33}$ | $X_2^3 + X_0^2X_1 + X_1^2X_3 + X_0X_3^2 + X_0X_1X_2 = 0$ |
| $F_{34}$ | $X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_0^2X_1 + X_0^2X_2 + X_0^2X_3 = 0$ |
| $F_{35}$ | $X_0^3 + X_2^3 + X_3^3 + X_0^2X_2 + X_0^2X_3 + X_1^2X_2 + X_0X_1X_2 = 0$ |
| $F_{36}$ | $X_0^3 + X_1^3 + X_2^3 + X_0^2X_1 + X_0^2X_2 + X_0^2X_3 + X_1^2X_2 + X_0X_3^2 + X_0X_1X_2 = 0$ |

set $\mathrm{PG}(19,2)$ of order $2^{20} - 1 = 1048575$. Using the fundamental quadrangle as base, the group has a strong generating set consisting of the following 6 generators:

$$s_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, s_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, s_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$s_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, s_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, s_6 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the method of Schreier trees will apply every generator to the every element of $\mathrm{PG}(19,2)$, there are $6 \times 1048575$ operations to consider. This can be done in reasonable time.

□

Note that in Table 1, the surfaces $F_{37}, \cdots F_{47}$ do not have any singular points over $\mathbb{F}_2$ but singular points for these surfaces appears over the extension fields. Therefore, we do not consider them as nonsingular cubic surfaces.

**Remark 3.5** *For instance, the surface $F_1$ over $\mathbb{F}_2$ given by the equation*

$$x_0^2x_1 + x_0x_1^2 + x_2^2x_3 + x_2x_3^2 = 0$$

*has all 15 points of $PG(3,2)$. The stabiliser of the surface equation differs from the stabiliser of the associated set of points. The stabiliser group of $PG(3,2)$ is $PGL(4,2)$ having order 20160 while the stabiliser group of the surface $F_1$ has order 720.*

**Remark 3.6** *Regarding an independent verification of Dickson' s list, we match the Dickson surfaces from Table 2 with the orbits that we got within the proof of Theorem 3.4 using the information contained in the Schreier trees in Table 3.*

We use the following notation: Let $N$ be the number of points of a surface. Let $m$ be the number of orbits of $\mathrm{PGL}(4,2)$ on subset of size $N$ ($m$ depends on $N$). For each orbit, pick a representative $S_j^N$ where $j \in \{1, \cdots, m\}$. $|Aut(S_j^N)|$ denotes the order of the projective stabiliser of $S_j^N$. Let $k$ be the number of orbits of $Aut(S_j^N)$ on the set of equations of cubic surfaces vanishing on $S_j^N$. Let $F_i$ be the orbit representative and $|Aut(F_i)|$ be the order of the projective stabiliser of $F_i$. Let $C_2^t$ be the number of nonsingular cubic surfaces with $q^2 + tq + 1$ points over $\mathbb{F}_2$.

In addition, the number of points, lines, singular points, eckardt points and double points for each surface can be found in the first column of the Table 1. $\#P$, $\#l$, $\#S$, $\#E, \#D$ denote the number of points, lines, singular points, Eckardt points and double points, respectively.

In the work of Dickson, the automorphism group of each surface is not available. In Table 3, we add the column of $|Aut(F_i)|$ for each nonsingular $F_i$. Moreover, we checked our classification result with the enumerative formula of Das. For instance; The number of nonsingular cubic surfaces with $q^2 + 2q + 1$ points over $\mathbb{F}_q$

$$\frac{(q^2-1)q^6(q^3-1)(q^4-1)(91q^4-5q^3+36q^2-35q-15)}{360}$$

by Das [10]. So, $C_2^2 = 82600$. $g = |\mathrm{PGL}(4,2)| = 20160$ from Equation 2.1. Using Equation 2.2 and the result in Table 3 (8th column) when $t = 2$,

$$20160 \cdot \left(\frac{1}{24} + \frac{1}{18} + \frac{1}{12} + \frac{1}{6} + \frac{1}{4} + \frac{1}{2} + \frac{3}{1}\right) = 82600.$$

**Table 3**. Nonsingular cubic surfaces over $\mathbb{F}_2$.

| $N$ | t | $m$ | $S_j^N$ | $\|Aut(S_j^N)\|$ | $k$ | $F_i$ | $\|Aut(F_i)\|$ | $C_2^t$ |
|---|---|---|---|---|---|---|---|---|
| 15 | 5 | 1 | $\{1,2,\dots,15\}$ | 20160 | 1 | $F_1$ | 720 | 28 |
| 13 | 4 | 1 | $\{3,4,\dots,15\}$ | 192 | 1 | $F_2$ | 12 | 1680 |
| 11 | 3 | 2 | $\{4,5,6,7,8,9,10,12,13,14,15\}$ | 192 | 1 | $F_3$ | 16 | 11340 |
| | | | $\{1,2,3,4,5,6,7,8,9,10,15\}$ | 24 | 1 | $F_4$ | 2 | |
| 9 | 2 | 5 | $\{4,7,9,10,11,12,13,14,15\}$ | 192 | 1 | $F_{23}$ | 24 | 82600 |
| | | | $\{7,8,9,10,11,12,13,14,15\}$ | 8 | 2 | $F_{11}, F_{13}$ | 1, 1 | |
| | | | $\{6,7,8,9,11,12,13,14,15\}$ | 72 | 2 | $F_{12}, F_8$ | 6, 18 | |
| | | | $\{6,7,8,9,10,11,12,14,15\}$ | 12 | 2 | $F_7, F_{10}$ | 2, 1 | |
| | | | $\{1,2,4,5,7,8,11,14,15\}$ | 48 | 2 | $F_5, F_6$ | 12, 4 | |
| 7 | 1 | 5 | $\{1,2,3,5,6,8,11\}$ | 1344 | 1 | $F_9$ | 48 | 129780 |
| | | | $\{1,2,3,4,5,6,8\}$ | 24 | 2 | $F_{15}, F_{16}$ | 2, 2 | |
| | | | $\{1,2,3,4,5,6,7\}$ | 48 | 2 | $F_{24}, F_{14}$ | 6,4 | |
| | | | $\{1,2,3,4,5,6,9\}$ | 8 | 3 | $F_{27}, F_{18}, F_{19}$ | 1, 1, 2 | |
| | | | $\{1,2,3,4,5,6,15\}$ | 8 | 3 | $F_{26}, F_{25}, F_{17}$ | 1, 2, 1 | |
| 5 | 0 | 4 | $\{1,2,3,5,6\}$ | 64 | 2 | $F_{32}, F_{21}$ | 4,4 | 84672 |
| | | | $\{1,2,3,4,5\}$ | 12 | 4 | $F_{31}, F_{30}, F_{29}, F_{20}$ | 2,1, 2, 2 | |
| | | | $\{1,2,3,4,15\}$ | 120 | 1 | $F_{28}$ | 5 | |
| | | | $\{1,2,3,4,11\}$ | 24 | 1 | $F_{33}$ | 1 | |
| 3 | -1 | 1 | $\{1,2,3\}$ | 48 | 1 | $F_{35}$ | 2 | 11620 |
| | | | $\{1,2,5\}$ | 576 | 2 | $F_{34}, F_{22}$ | 18, 48 | |
| 1 | -2 | 1 | $\{1\}$ | 1344 | 1 | $F_{36}$ | 24 | 840 |

## 4. Dickson surfaces over larger fields

In this section, we collect properties of nonsingular Dickson surfaces over small extension fields and make some observations. For the sake of simplicity, by nonsingular Dickson surface mean the surfaces $F_1, \dots, F_{36}$ which are nonsingular over $\mathbb{F}_2$.

Let $\mathcal{F}_k^i$ be the nonsingular Dickson cubic surface with $i$ lines over $\mathbb{F}_{2^k}$. In the following theorems we consider the properties of the nonsingular Dickson surfaces over small extension fields of $\mathbb{F}_2$. For each surface we count the number of lines, points, Eckardt points, singular points and double points. In addition, we found the order of the collineation stabiliser groups for each such surface over $\mathbb{F}_q$ where $q = 4$ and $q = 8$.

**Theorem 4.1** *Consider $\mathcal{F}_k^i$, the nonsingular Dickson cubic surface with $i$ lines over $\mathbb{F}_{2^k}$ where $k = 2, \dots, 6$. The possible $i$' are the following:*

*(a)* $27, 9, 7, 5, 3, 2, 1$ *or* $0$, *when* $k = 2$,

*(b)* $27, 15, 7, 5, 4, 3, 2, 1, 0$, *when* $k = 3$,

*(c)* $27, 9, 3, 2, 0$, *when* $k = 4$,

*(d)* $27, 15, 9, 5, 4, 3, 2, 1, 0$, *when* $k = 5$,

*(e)* $27, 7, 5, 3, 0$, *when* $k = 6$.

**Proof** We consider the nonsingular Dickson surfaces over the extension fields $\mathbb{F}_{2^k}$ of $\mathbb{F}_2$ where $2 \leq k \leq 6$. To count the number of lines, we used Orbiter[‖]. First, Orbiter computes the set of points on each surface, including the set of singular points. After that, Orbiter determines the set of lines on the surface. Using the intersection graph of lines, it then determines the double points and Eckardt points. Orbiter uses Nauty[**] to compute the automorphism groups using graph theory, as described in [1]. Because of the computational complexity, we are not able to compute the automorphism groups of surfaces over the fields $\mathbb{F}_{16}$, $\mathbb{F}_{32}$ and $\mathbb{F}_{64}$. □

Let $e, f, g$ and $h$ denote the number of Eckardt points, the number of double points, the number of single points, the number of points which is not on any line of the surface, respectively. Let us denote $(e, f, g, h)$ as the type of the associated cubic surface. These types are good to distinguish different surfaces but does not guarantee the equivalence. Two surface with the same type can be distinct. Further classification algorithms are needed.

**Theorem 4.2** *Let* $\mathcal{F}_2^i$ *be the nonsingular Dickson cubic surfaces with* $i$ *lines over* $\mathbb{F}_4$. *There are at least one of* $\mathcal{F}_2^{27}$, $3$ *of* $\mathcal{F}_2^9$, $2$ *of* $\mathcal{F}_2^7$, $4$ *of* $\mathcal{F}_2^3$, $2$ *of* $\mathcal{F}_2^2$, *one of* $\mathcal{F}_2^1$, $7$ *of* $\mathcal{F}_2^0$ *over* $\mathbb{F}_4$.

**Proof** $F_1$ and $F_9$ have 27 lines and 45 Eckardt points. These surfaces are equivalent because there is only one surface with 27 lines over $\mathbb{F}_4$. They belong to the Hirschfeld family. The cubic surfaces corresponding to $\mathcal{F}_2^9$ come in three types. There is one Dickson surface of type $(6, 0, 27, 0)$ with automorphism group order 432, four of the type $(4, 6, 21, 2)$ with automorphism group order 24 and two of the type $(1, 15, 12, 5)$ with automorphism group order 4. The cubic surfaces corresponding to $\mathcal{F}_2^7$ come in two types. There are two Dickson surfaces of type $(3, 0, 26, 0)$ with automorphism group order 384, one of the type $(1, 6, 20, 2)$ with automorphism group order 8. The cubic surfaces corresponding to $\mathcal{F}_2^3$ come in four types. There are two Dickson surfaces of type $(1, 0, 12, 16)$ with automorphism group order 12, one of the type $(1, 0, 12, 16)$ with automorphism group order 192, one of the type $(1, 0, 12, 16)$ with automorphism group order 8 and two of the type $(0, 3, 9, 17)$ with automorphism group order 2. The cubic surfaces corresponding to $\mathcal{F}_2^2$ come in two types. There are three Dickson surfaces of the type $(0, 0, 10, 15)$ with automorphism group order 2 and one of the type $(0, 0, 10, 15)$ with automorphism group order 20. The cubic surfaces corresponding to $\mathcal{F}_2^1$ come in one type. There are two Dickson surfaces of type $(0, 0, 5, 12)$. The cubic surfaces corresponding to $\mathcal{F}_2^0$ come in seven types. There is one Dickson surface of type $(0, 0, 0, 25)$ with automorphism group order 48, two of the type $(0, 0, 0, 25)$ with automorphism group order 4, one of the type $(0, 0, 0, 25)$ with automorphism group order 2, two of the type $(0, 0, 0, 21)$ with automorphism group order 4, two of the type $(0, 0, 0, 21)$ with automorphism group order 216, two of the type $(0, 0, 0, 21)$ with automorphism group order 2 and two of the type $(0, 0, 0, 9)$. □

**Remark 4.3** *The possibilities for the number of lines of a nonsingular cubic surface over a finite field of characteristic two was unknown. Dickson [11] finds examples with* $15, 9, 5, 3, 2, 1, 0$ *lines. Hirschfeld [15] finds example with 27 lines. Here, we give some examples with 7 lines, for instance, the cubic surfaces* $F_{22}, F_3$ *and* $F_4$ *over* $\mathbb{F}_4$ *have 7 lines. It is the very recent that Mckean also finds an example of nonsingular cubic surfaces with 7 lines over the field of characteristic two*[††].

[**] McKay (2020). Nauty and Traces (Version 2.7r1) [online]. Website https://users.cecs.anu.edu.au/bdm/nauty/

[††] Mckean S., Rational lines on smooth cubic surfaces, Preprint

**Theorem 4.4** *Let $\mathcal{F}_3^i$ be the nonsingular Dickson cubic surfaces with $i$ lines over $\mathbb{F}_8$. There are at least one of $\mathcal{F}_3^{27}$, 2 of $\mathcal{F}_3^{15}$, 2 of $\mathcal{F}_3^{7}$, 2 of $\mathcal{F}_3^{5}$, 8 of $\mathcal{F}_3^{3}$, one of $\mathcal{F}_3^{2}$, 5 of $\mathcal{F}_3^{1}$, 3 of $\mathcal{F}_3^{0}$ over $\mathbb{F}_8$.*

**Proof** $F_{36}$ and $F_2$ have 27 lines and 13 Eckardt points. These surfaces are isomorphic because there is only one surface with 27 lines over $\mathbb{F}_8$. They are members of the Tekirdağ 1 family described in[†]. The cubic surfaces corresponding to $\mathcal{F}_3^{15}$ come in two types. There are three Dickson surfaces of type $(15,0,90,0)$ with automorphism group order 2160 and one of the type $(1,42,48,14)$ with automorphism group order 6. The cubic surfaces corresponding to $\mathcal{F}_3^{7}$ come in two types. There is one Dickson surface of type $(3,0,54,32)$ with automorphism group order 48 and one of the type $(0,9,45,35)$ with automorphism group order 6. The cubic surfaces corresponding to $\mathcal{F}_3^{5}$ come in two types. There is one Dickson surface of type $(2,0,39,48)$ with automorphism group order 48 and one of the type $(1,3,36,49)$ with automorphism group order 6. The cubic surfaces corresponding to $\mathcal{F}_3^{3}$ come in eight types. There is one Dickson surface of type $(1,0,24,64)$ with automorphism group order 48, two of the type $(1,0,24,64)$ with automorphism group order 6, one of the type $(0,3,21,65)$ with automorphism group order 3, two of the type $(1,0,24,48)$ with automorphism group order 144, one of the type $(1,0,24,48)$ with automorphism group order 6, two of the type $(1,0,24,32)$ with automorphism group order 576, one of the type $(1,0,24,32)$ with automorphism group order 48 and one of the type $(0,3,21,33)$ with automorphism group order 6. The cubic surfaces corresponding to $\mathcal{F}_3^{2}$ come in one type. There are two Dickson surfaces of the type $(0,0,18,63)$ with automorphism group order 3. The cubic surfaces corresponding to $\mathcal{F}_3^{1}$ come in five types. There are three Dickson surfaces of the type $(0,0,9,64)$ with automorphism group order 6, two of the type $(0,0,9,64)$ with automorphism group order 3, one of the type $(0,0,9,64)$ with automorphism group order 12, two of the type $(0,0,9,48)$ with automorphism group order 144 and one of the type $(0,0,9,48)$ with automorphism group order 3. The cubic surfaces corresponding to $\mathcal{F}_3^{0}$ come in three types. There is one Dickson surface of type $(0,0,0,65)$ with automorphism group order 3, one of the type $(0,0,0,65)$ with automorphism group order 15 and two of the type $(0,0,0,49)$ with automorphism group order 3. □

**Theorem 4.5** *Let $\mathcal{F}_4^i$ be the nonsingular Dickson cubic surfaces with $i$ lines over $\mathbb{F}_{16}$. There are at least 2 of $\mathcal{F}_4^{27}$, 4 of $\mathcal{F}_4^{9}$, 2 of $\mathcal{F}_4^{3}$, one of $\mathcal{F}_4^{2}$, 2 of $\mathcal{F}_4^{0}$ over $\mathbb{F}_{16}$.*

**Proof** $F_1, F_{22}, F_9$ and $F_3$ have 27 lines and 45 Eckardt points. These surfaces are isomorphic over $\mathbb{F}_{16}$ but not over $\mathbb{F}_4$. They are the members of Hirschfeld family. $F_4$ has 27 lines and 5 Eckardt points. This is a member of the family with 5 Eckardt points described in[*]. The cubic surfaces corresponding to $\mathcal{F}_4^{9}$ come in four types. There are two Dickson surfaces of type $(6,0,135,180)$, four of the type $(4,6,129,182)$, two of the type $(1,15,120,185)$ and one of the type $(0,18,117,186)$. The cubic surfaces corresponding to $\mathcal{F}_4^{3}$ come in two types. There are four Dickson surfaces of type $(1,0,48,192)$ and two of the type $(0,3,45,193)$. The cubic surfaces corresponding to $\mathcal{F}_4^{2}$ come in one type. There are four Dickson surfaces of type $(0,0,34,255)$. The cubic surfaces corresponding to $\mathcal{F}_4^{0}$ come in two types. There are six Dickson surfaces of type $(0,0,0,273)$ and six of the type $(0,0,0,225)$. □

**Theorem 4.6** *Let $\mathcal{F}_5^i$ be the nonsingular Dickson cubic surfaces with $i$ lines over $\mathbb{F}_{32}$. There are at least 2 of $\mathcal{F}_5^{27}$, 2 of $\mathcal{F}_5^{15}$, one of $\mathcal{F}_5^{9}$, 2 of $\mathcal{F}_5^{5}$, 4 of $\mathcal{F}_5^{3}$, 4 of $\mathcal{F}_5^{1}$, 5 of $\mathcal{F}_5^{0}$ over $\mathbb{F}_{32}$.*

**Proof** $F_{11}$ and $F_{10}$ have 27 lines. They are not isomorphic. $F_{11}$ has 5 Eckardt points and belongs to the family with 5 Eckardt points described in*. The surface $F_{10}$ does not have any Eckardt points and it is a member of the family with 0 Eckardt point described in*. The cubic surfaces corresponding to $\mathcal{F}_5^{15}$ come in two types. There are two Dickson surfaces of type $(15, 0, 450, 720)$ and one of the type $(0, 45, 405, 735)$. The cubic surfaces corresponding to $\mathcal{F}_5^{9}$ come in one type. There is one Dickson surface of type $(4, 6, 273, 870)$. The cubic surfaces corresponding to $\mathcal{F}_5^{5}$ come in two types. There is one Dickson surface of type $(2, 0, 159, 960)$ and one of the type $(1, 3, 156, 961)$. The cubic surfaces corresponding to $\mathcal{F}_5^{3}$ come in four types. There are two Dickson surfaces of type $(1, 0, 96, 992)$, one of the type $(0, 3, 93, 993)$, one of the type $(0, 0, 99, 990)$ and one of the type $(1, 0, 96, 960)$. The cubic surfaces corresponding to $\mathcal{F}_5^{1}$ come in four types. There are two Dickson surfaces of type $(0, 0, 33, 1056)$, six of the type $(0, 0, 33, 1024)$, two of the type $(0, 0, 33, 992)$ and one of the type $(0, 0, 33, 960)$. The cubic surfaces corresponding to $\mathcal{F}_5^{0}$ come in five types. There is one Dickson surface of type $(0, 0, 0, 1089)$, four of the type $(0, 0, 0, 1057)$, four of the type $(0, 0, 0, 1025)$, two of the type $(0, 0, 0, 993)$ and one of the type $(0, 0, 0, 961)$. □

**Theorem 4.7** *Let $\mathcal{F}_6^{i}$ be the nonsingular Dickson cubic surfaces with $i$ lines over $\mathbb{F}_{64}$. There are at least 3 of $\mathcal{F}_6^{27}$, 3 of $\mathcal{F}_6^{7}$, 4 of $\mathcal{F}_6^{3}$, one of $\mathcal{F}_6^{2}$, one of $\mathcal{F}_6^{0}$ over $\mathbb{F}_{64}$.*

**Proof** $F_1, F_{34}, F_9, F_{24}, F_8$ have 27 lines and 45 Eckardt points. They are all isomorphic and belong to Hirschfeld family. $F_{36}, F_2, F_{21}, F_{23}, F_5$ and $F_6$ have 27 lines and 13 Eckardt points. $F_{35}, F_{20}, F_{25}$ and $F_7$ have 27 lines and 1 Eckardt point. They are members of the family with 1 Eckardt points over the field of characteristic two described in*. The cubic surfaces corresponding to $\mathcal{F}_6^{7}$ come in three types. There are three Dickson surfaces of type $(3, 0, 446, 3840)$, one of the type $(1, 6, 440, 3842)$ and one of the type $(0, 9, 437, 3843)$. The cubic surfaces corresponding to $\mathcal{F}_6^{3}$ come in four types. There are four Dickson surfaces of type $(1, 0, 192, 4096)$, two of the type $(0, 3, 189, 4097)$, three of the type $(1, 0, 192, 3840)$ and one of the type $(0, 3, 189, 3841)$. The cubic surfaces corresponding to $\mathcal{F}_6^{2}$ come in one type. There are four Dickson surfaces of type $(0, 0, 130, 4095)$. The cubic surfaces corresponding to $\mathcal{F}_6^{0}$ come in one type. There are two Dickson surfaces of type $(0, 0, 0, 3969)$. □

**Remark 4.8** *By†, there are 3 families of cubic surfaces with 27 lines and 13 Eckardt points over $\mathbb{F}_{64}$, namely Tekirdağ 1, Kapadokya 1 and 2. By Theorem 4.4, $F_{36}, F_2$ are belong to Tekirdağ 1. Using Orbiter's surface recognition, we confirm that $F_6$ is Tekirdağ 1 also. Unfortunately the recognition was unsuccesful for $F_5, F_{21}$ and $F_{23}$ (see Section 7, item 5).*

## 5. Properties of nonsingular cubic surfaces

The data collected in this work may be helpful to investigate properties of nonsingular cubic surfaces with less than 27 lines. We study the structure of these new examples. The properties of interest are:

1. the number of lines,
2. degeneration of Schläfli structure,
3. the relation between surface and a plane: Clebsch map,

4. the smallest field extension where all 27 lines appear,
5. new families with 27 lines,
6. automorphism groups,
7. isomorphism over extension fields.

### 5.1. The number of lines

The surface $F_1$ over $\mathbb{F}_2$ has 35 combinatorial lines but only 15 real lines (see Table 4).

**Table 4**. 27 Lines of $F_1$ in $\mathrm{PG}(3,4)$.

$$a_1 = \begin{bmatrix} 1 & 0 & \omega^2 & \omega^2 \\ 0 & 1 & 0 & \omega^2 \end{bmatrix}, a_2 = \begin{bmatrix} 1 & 0 & 0 & \omega \\ 0 & 1 & \omega & \omega \end{bmatrix}, a_3 = \begin{bmatrix} 1 & 0 & \omega & \omega \\ 0 & 1 & \omega & 0 \end{bmatrix}, a_4 = \begin{bmatrix} 1 & 0 & 0 & \omega^2 \\ 0 & 1 & \omega^2 & 0 \end{bmatrix},$$

$$a_5 = \begin{bmatrix} 1 & 0 & \omega & 0 \\ 0 & 1 & 0 & \omega \end{bmatrix}, a_6 = \begin{bmatrix} 1 & 0 & \omega^2 & 0 \\ 0 & 1 & \omega^2 & \omega^2 \end{bmatrix}, b_1 = \begin{bmatrix} 1 & 0 & \omega & \omega \\ 0 & 1 & 0 & \omega \end{bmatrix}, b_2 = \begin{bmatrix} 1 & 0 & 0 & \omega^2 \\ 0 & 1 & \omega^2 & \omega^2 \end{bmatrix},$$

$$b_3 = \begin{bmatrix} 1 & 0 & \omega^2 & \omega^2 \\ 0 & 1 & \omega^2 & 0 \end{bmatrix}, b_4 = \begin{bmatrix} 1 & 0 & 0 & \omega \\ 0 & 1 & \omega & 0 \end{bmatrix}, b_5 = \begin{bmatrix} 1 & 0 & \omega^2 & 0 \\ 0 & 1 & 0 & \omega^2 \end{bmatrix}, b_6 = \begin{bmatrix} 1 & 0 & \omega & 0 \\ 0 & 1 & \omega & \omega \end{bmatrix},$$

$$c_{12} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, c_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, c_{14} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, c_{15} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$c_{16} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, c_{23} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, c_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, c_{25} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

$$c_{26} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, c_{34} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, c_{35} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, c_{36} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$c_{45} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, c_{46} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, c_{56} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$F_3$ and $F_{22}$ are nonsingular over $\mathbb{F}_4$ with 7 lines each. This is a contrubition to the study of the possible number of lines of a nonsingular cubic surfaces over finite fields of characteristic two.

### 5.2. Degeneration of Schläfli structure

Recall Schläfli labelling. We describe similar structures for cubic surfaces with 15 and 9 lines which are the complement of the double-six and double-three, respectively.

**Example 5.1** *The surface $F_1$ has 15 points and 15 lines over $\mathbb{F}_2$. Every line has 3 points on it. There is neither a singular point nor a double point. However, there are 15 Eckardt points. Table 5 shows the lines of $F_1$ over $\mathbb{F}_2$.*

*The intersection table of these lines can be seen in Table 6.*

**Table 5**. 15 lines of $F_1$ in $\mathrm{PG}(3,2)$.

$$\ell_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \ell_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \ell_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \ell_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ell_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\ell_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ell_6 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \ell_7 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \ell_8 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \ell_9 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\ell_{10} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \ell_{11} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ell_{12} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \ell_{13} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \ell_{14} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

**Table 6**. Intersection table of the lines $\ell_i$.

| | $\ell_0$ | $\ell_1$ | $\ell_2$ | $\ell_3$ | $\ell_4$ | $\ell_5$ | $\ell_6$ | $\ell_7$ | $\ell_8$ | $\ell_9$ | $\ell_{10}$ | $\ell_{11}$ | $\ell_{12}$ | $\ell_{13}$ | $\ell_{14}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\ell_0$ | – | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\ell_1$ | 1 | – | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $\ell_2$ | 1 | 1 | – | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| $\ell_3$ | 1 | 0 | 0 | – | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\ell_4$ | 1 | 0 | 0 | 1 | – | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| $\ell_5$ | 0 | 0 | 1 | 1 | 0 | – | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| $\ell_6$ | 0 | 0 | 1 | 0 | 1 | 1 | – | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| $\ell_7$ | 1 | 0 | 0 | 0 | 0 | 1 | 0 | – | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| $\ell_8$ | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | – | 0 | 1 | 1 | 0 | 1 | 0 |
| $\ell_9$ | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | – | 1 | 0 | 1 | 1 | 0 |
| $\ell_{10}$ | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | – | 1 | 0 | 0 | 1 |
| $\ell_{11}$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | – | 1 | 0 | 0 |
| $\ell_{12}$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | – | 0 | 0 |
| $\ell_{13}$ | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | – | 1 |
| $\ell_{14}$ | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | – |

We are using the Schläfli labelling for the subset of 15 lines. Using the intersection properties of lines in their Schläfli labelling, the incidence between the 15 lines can be described in the following way: Every line meets 6 other and skew to 8. The 8 lines which are skew to $c_{12}$ form a double-four in the following sense: two lines are skew if and only if they are in the same row or same column of the following array.

$$\begin{matrix} c_{13} & c_{14} & c_{15} & c_{16} \\ c_{23} & c_{24} & c_{25} & c_{26} \end{matrix}$$

The remaining 6 lines arise from the intersection of plane pairs associated with the diagonals of all $2 \times 2$ submatrices of this array.

We match the lines from the Example 5.1 with the Schläfli labelling. It can be seen in Table 7.

**Example 5.2** *The surface $F_8$ has 33 points and 9 lines over $\mathbb{F}_4$, shown in Table 8. Every line has 5 points*

**Table 7**. Schläfli labels for 15 lines.

| Schläfli labelling | $\ell_i$ |
|---|---|
| $c_{12}$ | $\ell_0$ |
| $c_{13}$ | $\ell_5$ |
| $c_{14}$ | $\ell_{10}$ |
| $c_{15}$ | $\ell_{12}$ |
| $c_{16}$ | $\ell_{13}$ |

| Schläfli labelling | $\ell_i$ |
|---|---|
| $c_{23}$ | $\ell_9$ |
| $c_{24}$ | $\ell_6$ |
| $c_{25}$ | $\ell_{14}$ |
| $c_{26}$ | $\ell_{11}$ |
| $c_{34}$ | $\ell_1$ |

| Schläfli labelling | $\ell_i$ |
|---|---|
| $c_{35}$ | $\ell_8$ |
| $c_{36}$ | $\ell_4$ |
| $c_{45}$ | $\ell_3$ |
| $c_{46}$ | $\ell_7$ |
| $c_{56}$ | $\ell_2$ |

*on it. There is neither singular point nor double point. However there are* 6 *Eckardt points.* 27 *points of* $F_8$ *lie on a single line of* $F_8$*. So, all points of the surface lie on the lines of the surface.*

**Table 8**. 9 lines of $F_8$ in $\mathrm{PG}(3,4)$.

$$L_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad L_1 = \begin{bmatrix} 1 & 0 & \omega^2 & 0 \\ 0 & 1 & 0 & \omega^2 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & 0 & \omega & 0 \\ 0 & 1 & 0 & \omega \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad L_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad L_5 = \begin{bmatrix} 1 & 0 & \omega^2 & \omega^2 \\ 0 & 1 & \omega^2 & 0 \end{bmatrix}$$

$$L_6 = \begin{bmatrix} 1 & 0 & 0 & \omega^2 \\ 0 & 1 & \omega^2 & \omega^2 \end{bmatrix} \quad L_7 = \begin{bmatrix} 1 & 0 & \omega & \omega \\ 0 & 1 & \omega & 0 \end{bmatrix} \quad L_8 = \begin{bmatrix} 1 & 0 & 0 & \omega \\ 0 & 1 & \omega & \omega \end{bmatrix}$$

*The intersection table of these lines can be seen in Table 9.*

**Table 9**. Intersection table of the lines $L_i$.

| | $L_0$ | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | $L_6$ | $L_7$ | $L_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $L_0$ | - | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| $L_1$ | 0 | - | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| $L_2$ | 0 | 0 | - | 1 | 1 | 1 | 1 | 0 | 0 |
| $L_3$ | 0 | 1 | 1 | - | 0 | 0 | 1 | 0 | 1 |
| $L_4$ | 0 | 1 | 1 | 0 | - | 1 | 0 | 1 | 0 |
| $L_5$ | 1 | 0 | 1 | 0 | 1 | - | 0 | 0 | 1 |
| $L_6$ | 1 | 0 | 1 | 1 | 0 | 0 | - | 1 | 0 |
| $L_7$ | 1 | 1 | 0 | 0 | 1 | 0 | 1 | - | 0 |
| $L_8$ | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | - |

We are using the Schläfli labelling for the subset of 9 lines. Using the intersection properties of lines in their Schläfli labelling, the incidence between 9 lines can be described in the following way: Every line meets 4 and is skew to 4. The array of 6 lines

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}$$

form a double-three in the following sense: two lines are skew if and only if they are in the same row or same column.

The remanining 3 lines $c_{12}, c_{13}$ and $c_{23}$ arise from the intersection of plane pairs associated with the diagonals of all $2 \times 2$ submatrices of this array.

We match the lines from the Example 5.2 with the Schläfli labelling. It can be seen in Table 10.

**Table 10**. Schläfli labels for 9 lines.

| Schläfli labelling | $L_i$ |
|---|---|
| $a_1$ | $L_0$ |
| $a_2$ | $L_1$ |
| $a_3$ | $L_2$ |

| Schläfli labelling | $L_i$ |
|---|---|
| $b_1$ | $L_3$ |
| $b_2$ | $L_5$ |
| $b_3$ | $L_7$ |

| Schläfli labelling | $L_i$ |
|---|---|
| $c_{12}$ | $L_8$ |
| $c_{13}$ | $L_6$ |
| $c_{23}$ | $L_4$ |

### 5.3. The relation between surface and a plane

It is well known that there is a birational map from nonsingular cubic surface with 27 lines to a plane. The cubic surfaces with 27 lines are obtained by blowing-up of a plane at 6 points in general position. The relation between cubic surfaces with 27 lines and 6 points in a plane was observed by Clebsch [8]. For an explicit description, see [17]. Recall that nonsingular cubic surfaces $\mathcal{F}$ over a nonalgebraically closed field $\mathbb{F}$ may have less than 27 lines. However, the surface $\mathcal{F}$ would have all 27 lines over $\overline{\mathbb{F}}$. It means that some of the lines and points of $\mathcal{F}$ will appear over the extension fields of $\mathbb{F}$. Considering the birational map and base field we observe the following:

**Conjecture 5.3**
1. *A nonsingular cubic surface with 15 lines has $q^2 + 5q + 1$ points. There are 4 special skew lines where each of them maps to a single point in a plane.*
2. *A nonsingular cubic surface with 9 lines has $q^2 + 4q + 1$ points. There are 3 special skew lines where each of them maps to a single point in a plane.*

### 5.4. The smallest field extension where all 27 lines appear

20 of the Dickson surfaces have the property that all 27 lines appear in extension field of degree at most 6. For the surfaces $F_1$ and $F_9$, such field is $\mathbb{F}_4$. For the surfaces $F_{36}$ and $F_2$, such field is $\mathbb{F}_8$. For the surfaces $F_{22}$, $F_3$ and $F_4$ such field is $\mathbb{F}_{16}$. For the surfaces $F_{11}$ and $F_{10}$, such field is $\mathbb{F}_{32}$. For the surfaces $F_5, F_6, F_7, F_8, F_{20}, F_{21}, F_{23}, F_{24}, F_{25}, F_{34}$ and $F_{35}$, such field is $\mathbb{F}_{64}$.

**Example 5.4** *The surface $F_1$ has 27 lines over $\mathbb{F}_4$ but only 15 of them are real over $\mathbb{F}_2$. The lines can be seen in Table 4 where $\omega$ and $\omega^2$ are the elements of $\mathbb{F}_4$ which do not belong to $\mathbb{F}_2$.*

### 5.5. New families with 27 lines

**Corollary 5.5** *All families of cubic surfaces with 27 lines over fields of characteristic two have been described in*.*

1. *$F_1$ over $\mathbb{F}_4$ is a member of the Hirschfeld family.*
2. *$F_{36}$ over $\mathbb{F}_8$ is a member of the Tekirdağ 1 family described in†.*

3. *$F_4$ over $\mathbb{F}_{16}$ is a member of the family with 5 Eckardt points described in*.*

4. *$F_{10}$ over $\mathbb{F}_{32}$ is a member of the family with no Eckardt point described in*.*

5. *$F_{35}$ over $\mathbb{F}_{64}$ is a member of the family with one Eckardt point described in*.*

### 5.6. Automorphism groups

As pointed out in the introduction, the automorphism group of a surface can be different from the stabiliser of the set of points on the surface. This happens for surfaces with less than 19 points. Examples are $F_{13}$ and $F_{12}$ with 17 points over $\mathbb{F}_4$ and $F_{36}$ and $F_{23}$ with 9 points over $\mathbb{F}_4$. The orders of the set stabilisers of these surfaces are given in the following table. We do not know the order of the automorphism group of these surfaces (see Section 7, item 6).

| $F_i$ | $\|Aut(S_i)\|$ |
|---|---|
| $F_{36}$ | 384 |
| $F_{23}$ | 384 |
| $F_{13}$ | 2 |
| $F_{12}$ | 48 |

### 5.7. Isomorphism over extension Fields

Nonisomorphic surfaces over $\mathbb{F}_2$ may be isomorphic over an extention field. For instance, $F_1$ and $F_9$ are nonisomorphic over $\mathbb{F}_2$ but isomorphic over $\mathbb{F}_4$ (they are both Hirschfeld). In fact, the projectivity $\Upsilon$ given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & \omega & 1 \end{bmatrix}$$

maps $F_1$ to $F_9$.

## 6. Data set of cubic surfaces

The data associated with this paper is presented in the website ‖. The website contains a report for each $F_i$, both singular and nonsingular, where $i \in \{1, \ldots, 141\}$ over each field $\mathbb{F}_{2^k}$ where $k \leq 6$. For each $F_i$, the report includes the equation, the lines, the points, the singular points, the Eckardt points, the double points, the single points and the points not on any line. Besides this, information about how lines intersect each other is given. In order to use this data, the Orbiter labelling of field elements, points and lines in projective space needs to be considered. Orbiter labelling uses integers. Using integers is more concise than using polynomials. The Orbiter labelling can be found in User's guide‡‡. The website offers cheat sheets for all relevant finite fields and the 3-dimensional projective spaces.

## 7. Future works

The following problems remain for future work:

1. Find a better algorithm to classify cubic surfaces over finite fields.

‡‡Betten(2021). Orbiter User's Guide, Website https://www.math.colostate.edu/betten/orbiter/users_guide.pdf.

2. The relation induced by the birational structure between nonsingular cubic surfaces with less than 27 lines and the plane can be studied further.

3. Is there a relation between nonsingular cubic surfaces with 15 and 9 lines and 4-arcs and 3-arcs in the plane, respectively? Can this lead to a new and efficient clasification algorithm for cubic surfaces with 15 and 9 lines?

4. Determine the number of nonsingular cubic surfaces with 15 and 9 lines over a finite field.

5. Sort out the isomomorphism types of cubic surfaces with 13 Eckardt points over $\mathbb{F}_{64}$. Speciffically, determine which are Tekirdağ 1 and which are Kapadokya 1/2.

6. Determine the relation between the set stabiliser of the set of points on a surface and the stabiliser of the equation of the surface.

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