Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 753171, 11 pages
http://dx.doi.org/10.1155/2013/753171

Research Article

Existence and Uniqueness of Positive Solutions of Boundary-Value Problems for Fractional Differential Equations with p-Laplacian Operator and Identities on the Some Special Polynomials

Erdoğan Şen,[1,2] Mehmet Acikgoz,[3] Jong Jin Seo,[4,5] Serkan Araci,[6] and Kamil Oruçoğlu[2]

[1] Department of Mathematics, Faculty of Science and Letters, Namik Kemal University, 59030 Tekirdağ, Turkey
[2] Department of Mathematics Engineering, Istanbul Technical University, Maslak, 34469 Istanbul, Turkey
[3] Department of Mathematics, Faculty of Science and Arts, University of Gaziantep, 27310 Gaziantep, Turkey
[4] Department of Applied Mathematics, Pukyong National University, Busan 608-737, Republic of Korea
[5] Department of Mathematics, College of Natural Sciences, Kwangwoon University, Seoul 139-701, Republic of Korea
[6] Atatürk Street, 31290 Hatay, Turkey

Correspondence should be addressed to Serkan Araci; mtsrkn@hotmail.com

Received 13 May 2013; Accepted 28 September 2013

Academic Editor: Josip E. Pečarić

We consider the following boundary-value problem of nonlinear fractional differential equation with p-Laplacian operator $D^{\beta}_{0+}(\phi_p(D^{\alpha}_{0+}u(t))) + a(t)f(u) = 0$, $0 < t < 1$, $u(0) = \gamma u(h) + \lambda$, $u'(0) = \mu$, $\phi_p(D^{\alpha}_{0+}u(0)) = (\phi_p(D^{\alpha}_{0+}u(1)))' = (\phi_p(D^{\alpha}_{0+}u(0)))'' = (\phi_p(D^{\alpha}_{0+}u(0)))''' = 0$, where $1 < \alpha \leqslant 2$, $3 < \beta \leqslant 4$ are real numbers, $D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the standard Caputo fractional derivatives, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, $0 \leqslant \gamma < 1$, $0 \leqslant h \leqslant 1$, $\lambda, \mu > 0$ are parameters, $a : (0, 1) \to [0, +\infty)$, and $f : [0, +\infty) \to [0, +\infty)$ are continuous. By the properties of Green function and Schauder fixed point theorem, several existence and nonexistence results for positive solutions, in terms of the parameters λ and μ are obtained. The uniqueness of positive solution on the parameters λ and μ is also studied. In the final section of this paper, we derive not only new but also interesting identities related special polynomials by which Caputo fractional derivative.

1. Introduction

In 1695, L'Hôpital asked Leibniz: what if the order of the derivative is 1/2? To which Leibniz considered in a useful means, thus it follows that will be equal to $x\sqrt{dx : x}$, an obvious paradox. In recent years, fractional calculus has been studied by many mathematicians from Leibniz's time to the present.

Also, fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, fluid flows, electrical networks, viscoelasticity, aerodynamics, and many other branches of science. For details, see [1–9].

In the last few decades, fractional-order models are found to be more adequate than integer-order models for some real world problems. Recently, there have been some papers dealing with the existence and multiplicity of solutions (or positive solutions) of nonlinear initial fractional differential equations by the use of techniques of nonlinear analysis [10–21], upper and lower solutions method [22–24], fixed point index [25, 26], coincidence theory [27], Banach contraction mapping principle [28], and so forth.

Chai [11] investigated the existence and multiplicity of positive solutions for a class of boundary-value problem of fractional differential equation with p-Laplacian operator

$$D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right)+f\left(t,u(t),D_{0+}^{\rho}u(t)\right)=0,\quad 0<t<1,$$
$$u(0)=0,\qquad u(1)+\sigma D_{0+}^{\gamma}u(1)=0,\qquad D_{0+}^{\alpha}u(0)=0,\tag{1}$$

where $1<\alpha\leq 2$, $0<\gamma\leq 1$, $0\leq\alpha-\gamma-1$, σ is a positive constant number, and $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\gamma}$ are the standard Riemann-Liouville derivatives. By means of the fixed point theorem on cones, some existence and multiplicity results of positive solutions are obtained.

Although the fractional differential equation boundary-value problems have been studied by several authors, very little is known in the literature on the existence and nonexistence of positive solutions of fractional differential equation boundary-value problems with p-Laplacian operator when a parameter λ is involved in the boundary conditions. We also mention that, there is very little known about the uniqueness of the solution of fractional differential equation boundary-value problems with p-Laplacian operator on the parameter λ. Han et al. [29] studied the existence and uniqueness of positive solutions for the fractional differential equation with p-Laplacian operator

$$\begin{aligned}&D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right)+a(t)f(u)=0,\quad 0<t<1,\\&u(0)=\gamma u(\xi)+\lambda,\\&\phi_p\left(D_{0+}^{\alpha}u(0)\right)=\left(\phi_p\left(D_{0+}^{\alpha}u(1)\right)\right)'\\&\quad=\left(\phi_p\left(D_{0+}^{\alpha}u(0)\right)\right)''=0,\end{aligned}\tag{2}$$

where $0<\alpha\leqslant 1$, $2<\beta\leqslant 3$ are real numbers; $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Caputo fractional derivatives; $\phi_p(s)=|s|^{p-2}s$, $p>1$. Therefore, to enrich the theoretical knowledge of the above, in this paper, we investigate the following p-Laplacian fractional differential equation boundary-value problem:

$$\begin{aligned}&D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right)+a(t)f(u)=0,\quad 0<t<1,\\&u(0)=\gamma u(h)+\lambda,\qquad u'(0)=\mu,\\&\phi_p\left(D_{0+}^{\alpha}u(0)\right)=\left(\phi_p\left(D_{0+}^{\alpha}u(1)\right)\right)'=\left(\phi_p\left(D_{0+}^{\alpha}u(0)\right)\right)''\\&\quad=\left(\phi_p\left(D_{0+}^{\alpha}u(0)\right)\right)'''=0,\end{aligned}\tag{3}$$

where $1<\alpha\leqslant 2$, $3<\beta\leqslant 4$ are real numbers, $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Caputo fractional derivatives, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_p^{-1}=\phi_q$, $1/p+1/q=1$, $0\leqslant\gamma<1$, $0\leqslant h\leqslant 1$, $\lambda,\mu>0$ are parameters, $a:(0,1)\to[0,+\infty)$, and $f:[0,+\infty)\to[0,+\infty)$ are continuous. By the properties of Green function and Schauder fixed point theorem, several existence and nonexistence results for positive solutions, in terms of the parameters λ and μ are obtained. The uniqueness of positive solution on the parameters λ and μ, is also studied.

2. Preliminaries and Related Lemmas

Definition 1 (see [4]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $y:(0,+\infty)\to\mathbb{R}$ is given by

$$I_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds\tag{4}$$

provided the right side is pointwise defined on $(0,+\infty)$.

Definition 2 (see [4]). The Caputo fractional derivative of order $\alpha>0$ of a continuous function $y:(0,+\infty)\to\mathbb{R}$ is given by

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,\tag{5}$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $(0,+\infty)$.

Remark 3 (see [8]). By Definition 2, under natural conditions on the function $f(t)$, for $\alpha\to n$, the Caputo derivative becomes a conventional nth derivative of the function $f(t)$.

Remark 4 (see [4]). As a basic example,

$$\begin{aligned}D_{0^+}^{\alpha}t^{\mu}&=\mu(\mu-1)\cdots(\mu-n+1)\\&\quad\times\frac{\Gamma(1+\mu-n)}{\Gamma(1+\mu-\alpha)}t^{\mu-\alpha},\quad\text{for } t\in(0,\infty).\end{aligned}\tag{6}$$

In particular, $D_{0^+}^{\alpha}t^{\mu}=0$, $\mu=0,1,\ldots,n-1$, where $D_{0^+}^{\alpha}$ is the Caputo fractional derivative, n is the smallest integer greater than or equal to α.

From the definition of the Caputo derivative and Remark 4, we can obtain the following statement.

Lemma 5 (see [4]). *Let $\alpha>0$. Then, the fractional differential equation*

$$D_{0+}^{\alpha}u(t)=0\tag{7}$$

has a unique solution

$$\begin{aligned}u(t)&=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},\\&c_i\in\mathbb{R},\ i=0,1,2,\ldots,n-1,\end{aligned}\tag{8}$$

where n is the smallest integer greater than or equal to α.

Lemma 6 (see [4]). *Let $\alpha>0$. Assume that $u\in C^n[0,1]$. Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},\tag{9}$$

for some $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$, where n is the smallest integer greater than or equal to α.

Lemma 7. *Let $y\in C[0,1]$ and $1<\alpha\leq 2$. Then, fractional differential equation boundary-value problem*

$$D_{0+}^{\alpha}u(t)=y(t),\quad 0<t<1,\tag{10}$$

$$u(0)=\gamma u(h)+\lambda,\qquad u'(0)=\mu\tag{11}$$

has a unique solution

$$u(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{\gamma}{1-\gamma}\int_0^h\frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{\lambda+\gamma\mu h}{1-\gamma}. \tag{12}$$

Proof. We apply Lemma 6 to reduce (10) to an equivalent integral equation,

$$u(t)=I_{0+}^{\alpha}y(t)+c_0+c_1t,\quad c_0,c_1\in\mathbb{R}. \tag{13}$$

Consequently, the general solution of (10) is

$$u(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+c_0+c_1t,\quad c_0,c_1\in\mathbb{R}. \tag{14}$$

By (11), we have

$$c_0=\frac{\gamma}{1-\gamma}\int_0^h\frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{\gamma c_1 h}{1-\gamma}+\frac{\lambda}{1-\gamma}, \tag{15}$$

and since $u'(0)=c_1$, we have by (11)

$$c_1=\mu. \tag{16}$$

Therefore, the unique solution of problem (10) and (11) is

$$u(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{\gamma}{1-\gamma}\int_0^h\frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{\lambda}{1-\gamma}+\frac{\gamma\mu h}{1-\gamma}. \tag{17}$$

□

Lemma 8. *Let $y\in C[0,1]$ and $1<\alpha\le 2$, $3<\beta\le 4$. Then, fractional differential equation boundary-value problem*

$$D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right)+y(t)=0,\quad 0<t<1, \tag{18}$$

$$\begin{gathered}u(0)=\gamma u(h)+\lambda,\qquad u'(0)=\mu,\\ \phi_p\left(D_{0+}^{\alpha}u(0)\right)=\left(\phi_p\left(D_{0+}^{\alpha}u(1)\right)\right)'=\left(\phi_p\left(D_{0+}^{\alpha}u(0)\right)\right)''\\ =\left(\phi_p\left(D_{0+}^{\alpha}u(0)\right)\right)'''=0,\end{gathered} \tag{19}$$

has a unique solution

$$\begin{aligned}u(t)={}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1H(s,\tau)y(\tau)\,d\tau\right)ds\\&+\frac{\gamma}{1-\gamma}\int_0^h\frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1H(s,\tau)y(\tau)\,d\tau\right)ds\\&+\frac{\lambda+\gamma\mu h}{1-\gamma},\end{aligned} \tag{20}$$

where

$$H(t,s)=\begin{cases}\dfrac{t(\beta-1)(1-s)^{\beta-2}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0\le s\le t\le 1,\\[2mm] \dfrac{t(\beta-1)(1-s)^{\beta-2}}{\Gamma(\beta)}, & 0\le t\le s\le 1.\end{cases} \tag{21}$$

Proof. From Lemma 6, the boundary-value problems (18) and (19) are equivalent to the integral equation

$$\phi_p\left(D_{0+}^{\alpha}u(t)\right)=-I_{0+}^{\beta}y(t)+c_0+c_1t+c_2t^2+c_3t^3, \tag{22}$$

for some $c_0,c_1,c_2,c_3\in\mathbb{R}$; that is,

$$\phi_p\left(D_{0+}^{\alpha}u(t)\right)=-\int_0^t\frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\,d\tau+c_0+c_1t+c_2t^2+c_3t^3. \tag{23}$$

By the boundary conditions $\phi_p(D_{0+}^{\alpha}u(0))=(\phi_p(D_{0+}^{\alpha}u(1)))'=(\phi_p(D_{0+}^{\alpha}u(0)))''=(\phi_p(D_{0+}^{\alpha}u(0)))'''=0$, we have

$$c_0=c_2=c_3=0,\qquad c_1=\int_0^1\frac{(\beta-1)(1-\tau)^{\beta-2}}{\Gamma(\beta)}y(\tau)\,d\tau. \tag{24}$$

Therefore, the solution $u(t)$ of fractional differential equation boundary-value problems (18) and (19) satisfies

$$\begin{aligned}\phi_p\left(D_{0+}^{\alpha}u(t)\right)&=-\int_0^t\frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)\,d\tau\\&\quad+t\int_0^1\frac{(\beta-1)(1-\tau)^{\beta-2}}{\Gamma(\beta)}y(\tau)\,d\tau\\&=\int_0^1H(t,\tau)y(\tau)\,d\tau.\end{aligned} \tag{25}$$

Consequently, $D_{0+}^{\alpha}u(t)=\phi_q(\int_0^1H(t,\tau)y(\tau)d\tau)$. Thus, fractional differential equation boundary-value problem (18) and (19) is equivalent to the problem

$$\begin{gathered}D_{0+}^{\alpha}u(t)=\phi_q\left(\int_0^1H(t,\tau)y(\tau)\,d\tau\right),\quad 0<t<1,\\ u(0)=\gamma u(h)+\lambda,\quad u'(0)=\mu.\end{gathered} \tag{26}$$

Lemma 7 implies that the fractional differential equation boundary-value problems (18) and (19) have a unique solution,

$$\begin{aligned}u(t)={}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1H(s,\tau)y(\tau)\,d\tau\right)ds\\&+\frac{\gamma}{1-\gamma}\int_0^h\frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1H(s,\tau)y(\tau)\,d\tau\right)ds\\&+\frac{\lambda+\gamma\mu h}{1-\gamma}.\end{aligned} \tag{27}$$

The proof is complete. □

Lemma 9 (see [15]). *Let $1 < \alpha \leqslant 2, 3 < \beta \leqslant 4$. The function $H(t,s)$ is continuous on $[0,1] \times [0,1]$ and satisfies*

(1) $H(t,s) \geqslant 0$, $H(t,s) \leqslant H(1,s)$, *for* $t, s \in [0,1]$;

(2) $H(t,s) \geqslant t^{\beta-1} H(1,s)$, *for* $t, s \in (0,1)$.

Lemma 10 (Schauder fixed point theorem [30]). *Let (E,d) be a complete metric space, U be a closed convex subset of E, and $A : U \to U$ be a mapping such that the set $\{Au : u \in U\}$ is relatively compact in E. Then, A has at least one fixed point.*

To prove our main results, we use the following assumptions.

(H1) $0 < \int_0^1 H(1,\tau) a(\tau) d\tau < +\infty$;

(H2) there exist $0 < \sigma < 1$ and $c > 0$ such that

$$f(x) \leqslant \sigma L \phi_p(x), \quad \text{for } 0 \leqslant x \leqslant c, \tag{28}$$

where L satisfies

$$0 < L \leqslant \left[\phi_p \left(\frac{1 + \gamma (h^\alpha - 1)}{\Gamma(\alpha+1)(1-\gamma)} \right) \int_0^1 H(1,\tau) a(\tau) d\tau \right]^{-1}; \tag{29}$$

(H3) there exist $d > 0$ such that

$$f(x) \leqslant M \phi_p(x), \quad \text{for } d < x < +\infty, \tag{30}$$

where M satisfies

$$0 < M < \left[\phi_p \left(\frac{1 + \gamma (h^\alpha - 1)}{\Gamma(\alpha+1)(1-\gamma)} 2^{q-1} \right) \int_0^1 H(1,\tau) a(\tau) d\tau \right]^{-1}; \tag{31}$$

(H4) there exist $0 < \delta < 1$ and $e > 0$ such that

$$f(x) \geqslant N \phi_p(x), \quad \text{for } e < x < +\infty, \tag{32}$$

where N satisfies

$$N > \left[\phi_p \left(c_\delta \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} \phi_q \left(s^{\beta-1} \right) ds \right) \times \int_\delta^1 H(1,\tau) a(\tau) d\tau \right]^{-1}; \tag{33}$$

with

$$c_\delta = \int_0^\delta \alpha (1-s)^{\alpha-2} \phi_q \left(s^{\beta-1} \right) ds \in (0,1); \tag{34}$$

(H5) $f(x)$ is nondecreasing in x;

(H6) there exist $0 \leqslant \theta < 1$ such that

$$f(kx) \geqslant \left(\phi_p(k) \right)^\theta f(x), \quad \text{for any } 0 < k < 1,\ 0 < x < +\infty. \tag{35}$$

Remark 11. Let

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{\phi_p(x)}, \qquad f_\infty = \lim_{x \to +\infty} \frac{f(x)}{\phi_p(x)}. \tag{36}$$

Then, (H2) holds if $f_0 = 0$, (H3) holds if $f_\infty = 0$, and (H4) holds if $f_\infty = +\infty$.

3. Existence

Theorem 12. *Assume that (H1), (H2) hold. Then, the fractional differential equation boundary-value problem* (3) *has at least one positive solution for* $0 < \lambda + \gamma\mu \leq (1-\gamma)(1-\phi_q(\sigma))c$.

Proof. Let $c > 0$ be given in (H2). Define

$$K_1 = \{ u \in C[0,1] : 0 \leqslant u(t) \leqslant c \text{ on } [0,1] \} \tag{37}$$

and an operator $T_\lambda : K_1 \to C[0,1]$ by

$$\begin{aligned} T_\lambda u(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(s,\tau) a(\tau) f(u(\tau)) d\tau \right) \\ & + \frac{\gamma}{1-\gamma} \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \\ & \times \left(\int_0^1 H(s,\tau) a(\tau) f(u(\tau)) d\tau \right) ds \\ & + \frac{\lambda + \gamma\mu h}{1-\gamma}. \end{aligned} \tag{38}$$

Then, K_1 is a closed convex set. From Lemma 8, u is a solution of fractional differential equation boundary-value problem (3) if and only if u is a fixed point of T_λ. Moreover, a standard argument can be used to show that T_λ is compact.

For any $u \in K_1$, from (28) and (29), we obtain

$$\begin{gathered} f(u(t)) \leqslant \sigma L \phi_p(u(t)) \leqslant \sigma L \phi_p(c), \quad \text{on } [0,1], \\ \frac{1 + \gamma(h^\alpha - 1)}{\Gamma(\alpha+1)(1-\gamma)} \phi_q(L) \phi_q \left(\int_0^1 H(1,\tau) a(\tau) d\tau \right) \leqslant 1. \end{gathered} \tag{39}$$

Let $0 < \lambda + \gamma\mu \leqslant (1-\gamma)(1-\phi_q(\sigma))c$. Then, from Lemma 6 and (38), it follows that

$$\begin{aligned} 0 \leqslant & T_\lambda u(t) \\ \leqslant & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(1,\tau) a(\tau) f(u(\tau)) d\tau \right) ds \\ & + \frac{\gamma}{1-\gamma} \\ & \times \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 H(1,\tau) a(\tau) f(u(\tau)) d\tau \right) ds \\ & + \frac{\lambda + \gamma\mu h}{1-\gamma} \end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{\Gamma(\alpha+1)}\phi_q\left(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\right)\\
&\quad+\frac{\gamma h^\alpha}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\\
&\quad\times\left(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\right)+\left(1-\phi_q(\sigma)\right)c\\
&=\frac{1+\gamma h^\alpha-\gamma}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\left(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\right)\\
&\quad+\left(1-\phi_q(\sigma)\right)c\\
&\leqslant\frac{1+\gamma(h^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q(L)\phi_q\left(\int_0^1 H(1,\tau)a(\tau)d\tau\right)\phi_q(\sigma)c\\
&\quad+\left(1-\phi_q(\sigma)\right)c\\
&\leqslant\phi_q(\sigma)c+\left(1-\phi_q(\sigma)\right)c=c,\quad t\in[0,1].
\end{aligned}\tag{40}$$

Thus, $T_\lambda(K_1)\subseteq K_1$, by Schauder fixed point theorem, T_λ has a fixed point $u\in K_1$; that is, the fractional differential equation boundary-value problem (3) has at least one positive solution. The proof is complete. □

Corollary 13. *Assume that (H1) holds and $f_0 = 0$. Then, the fractional differential equation boundary-value problem* (3) *has at least one positive solution for sufficiently small $\lambda > 0$.*

Theorem 14. *Assume that (H1), (H3) hold. Then, the fractional differential equation boundary-value problem* (3) *has at least one positive solution for all $\lambda > 0$.*

Proof. Let $\lambda > 0$ be fixed and $d > 0$ be given in (H3). Define $D=\max_{0\leqslant x\leqslant d} f(x)$. Then

$$f(x)\leqslant D,\quad \text{for } 0\leqslant x\leqslant d. \tag{41}$$

From (31), we have

$$\frac{1+\gamma(h^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}2^{q-1}\phi_q(M)\phi_q\left(\int_0^1 H(1,\tau)a(\tau)d\tau\right)<1. \tag{42}$$

Thus, there exists $d^*>d$ large enough so that

$$\begin{aligned}
&\frac{1+\gamma(h^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}2^{q-1}\left(\phi_q(D)+\phi_q(M)d^*\right)\phi_q\\
&\quad\times\left(\int_0^1 H(1,\tau)a(\tau)d\tau\right)+\frac{\lambda+\gamma\mu h}{1-\gamma}\leqslant d^*.
\end{aligned}\tag{43}$$

Let

$$K_2=\{u\in C[0,1]:0\leqslant u(t)\leqslant d^* \text{ on } [0,1]\}. \tag{44}$$

For $u\in K_2$, define

$$\begin{aligned}
I_1^u&=\{t\in[0,1]:0\leqslant u(t)\leqslant d\},\\
I_2^u&=\{t\in[0,1]:d<u(t)\leqslant d^*\}.
\end{aligned}\tag{45}$$

Then, $I_1^u\cup I_2^u=[0,1]$, $I_1^u\cap I_2^u=\emptyset$, and in view of (30), we have

$$f(u(t))\leqslant M\phi_p(u(t))\leqslant M\phi_p(d^*),\quad \text{for } t\in I_2^u. \tag{46}$$

Let the compact operator T_λ be defined by (38). Then, from Lemma 6, (30), and (41), we have

$$\begin{aligned}
&0\leqslant T_\lambda u(t)\\
&\leqslant\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\right)ds\\
&\quad+\frac{\gamma}{1-\gamma}\\
&\quad\times\int_0^h\frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\right)ds\\
&\quad+\frac{\lambda+\gamma\mu h}{1-\gamma}\\
&\leqslant\frac{1}{\Gamma(\alpha+1)}\phi_q\left(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\right)\\
&\quad+\frac{\gamma h^\alpha}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\left(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\right)\\
&\quad+\frac{\lambda+\gamma\mu h}{1-\gamma}\\
&=\frac{1+\gamma(h^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\\
&\quad\times\left(\int_{I_1^u}H(1,\tau)a(\tau)f(u(\tau))d\tau\right.\\
&\qquad\left.+\int_{I_2^u}H(1,\tau)a(\tau)f(u(\tau))d\tau\right)\\
&\quad+\frac{\lambda+\gamma\mu h}{1-\gamma}\\
&\leqslant\frac{1+\gamma(h^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\\
&\quad\times\left(D\int_{I_1^u}H(1,\tau)a(\tau)d\tau\right.\\
&\qquad\left.+M\phi_p(d^*)\int_{I_2^u}H(1,\tau)a(\tau)d\tau\right)\\
&\quad+\frac{\lambda+\gamma\mu h}{1-\gamma}\\
&\leqslant\frac{1+\gamma(h^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\left(D+M\phi_p(d^*)\right)\phi_q\\
&\quad\times\left(\int_0^1 H(1,\tau)a(\tau)d\tau\right)+\frac{\lambda+\gamma\mu h}{1-\gamma}.
\end{aligned}\tag{47}$$

From (43) and the inequality $(a+b)^r \leqslant 2^r(a^r+b^r)$ for any $a,b,r>0$ (see, e.g., [31]), we obtain

$$\begin{aligned}0 &\leqslant T_\lambda u(t)\\ &\leqslant \frac{1+\gamma(h^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}2^{q-1}\left(\phi_q(D)+\phi_q(M)d^*\right)\phi_q\\ &\quad\times\left(\int_0^1 H(1,\tau)a(\tau)\,d\tau\right)+\frac{\lambda+\gamma\mu h}{1-\gamma}\leqslant d^*.\end{aligned} \tag{48}$$

Thus, $T_\lambda : K_2 \to K_2$. Consequently, by Schauder fixed point theorem, T_λ has a fixed point $u \in K_2$, that is, the fractional differential equation boundary-value problem (3) has at least one positive solution. The proof is complete. □

Corollary 15. *Assume that (H1) holds and $f_\infty = 0$. Then, the fractional differential equation boundary-value problem* (3) *has at least one positive solution for all $\lambda > 0$.*

4. Uniqueness

Definition 16 (see [32]). A cone P in a real Banach space X is called solid if its interior P^o is not empty.

Definition 17 (see [32]). Let P be a solid cone in a real Banach space X, $T : P^o \to P^o$ be an operator, and $0 \leqslant \theta < 1$. Then, T is called a θ-concave operator if

$$T(ku) \geqslant k^\theta Tu \quad \text{for any } 0<k<1,\ u\in P^o. \tag{49}$$

Lemma 18 (see [32, Theorem 2.2.6]). *Assume that P is a normal solid cone in a real Banach space X, $0 \leqslant \theta < 1$, and $T : P^o \to P^o$ is a θ-concave increasing operator. Then, T has only one fixed point in P^o.*

Theorem 19. *Assume that (H1), (H5), (H6) hold. Then, the fractional differential equation boundary-value problem* (3) *has a unique positive solution for any $\lambda > 0$.*

Proof. Define $P = \{u \in C[0,1] : u(t) \geqslant 0 \text{ on } [0,1]\}$. Then, P is a normal solid cone in $C[0,1]$ with

$$P^o = \{u\in C[0,1] : u(t) > 0 \text{ on } [0,1]\}. \tag{50}$$

For any fixed $\lambda > 0$, let $T_\lambda : P \to C[0,1]$ be defined by (38). Define $T : P \to C[0,1]$ by

$$\begin{aligned}Tu(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)ds\\ &\quad+\frac{\gamma}{1-\gamma}\int_0^h\frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\\ &\quad\times\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)ds.\end{aligned} \tag{51}$$

Then, from (H5), we have T increasing in $u \in P^o$ and

$$T_\lambda u(t) = Tu(t)+\frac{\lambda+\gamma\mu h}{1-\gamma}. \tag{52}$$

Clearly, $T_\lambda : P^o \to P^o$. Next, we prove that T_λ is a θ-concave increasing operator. In fact, for $u_1, u_2 \in P$ with $u_1(t) \geqslant u_2(t)$ on $[0,1]$, we obtain

$$T_\lambda u_1(t) \geqslant Tu_2(t)+\frac{\lambda+\gamma\mu h}{1-\gamma} = T_\lambda u_2(t); \tag{53}$$

that is, T_λ is increasing. Moreover, (H6) implies

$$\begin{aligned}T_\lambda(ku)(t) &\geqslant k^\theta\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\\ &\quad\times\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)ds\\ &\quad+k^\theta\frac{\gamma}{1-\gamma}\\ &\quad\times\int_0^h\frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)}\\ &\quad\times\phi_q\left(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))\,d\tau\right)ds\\ &\quad+\frac{\lambda+\gamma\mu h}{1-\gamma}\\ &= k^\theta Tu(t)+\frac{\lambda+\gamma\mu h}{1-\gamma}\\ &\geqslant k^\theta\left(Tu(t)+\frac{\lambda+\gamma\mu h}{1-\gamma}\right)=k^\theta T_\lambda u(t),\end{aligned} \tag{54}$$

that is, T_λ is θ-concave. By Lemma 8, T_λ has a unique fixed point u_λ in P^o, that is, the fractional differential equation boundary-value problem (3) has a unique positive solution. The proof is complete. □

5. Nonexistence

In this section, we let the Banach space $C[0,1]$ be endowed with the norm $\|u\| = \max_{0\leqslant t\leqslant 1}|u(t)|$.

Lemma 20. *Assume (H1) holds and let $0 < \delta < 1$ be given in (H4). Then, the unique solution $u(t)$ of fractional differential equation boundary-value problem* (18) *and* (19) *satisfies*

$$u(t) \geqslant c_\delta\|u\| \quad \textit{for } \delta\leqslant t\leqslant 1, \tag{55}$$

where c_δ is defined by (34).

Proof. In view of Lemma 9 and (19), we have

$$\begin{aligned}
u(t) \leqslant & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(1,\tau)\, y(\tau)\, d\tau\right) ds \\
& + \frac{\gamma}{1-\gamma} \\
& \times \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(s,\tau)\, y(\tau)\, d\tau\right) ds \\
& + \frac{\lambda+\gamma\mu h}{1-\gamma} \\
\leqslant & \frac{1}{\Gamma(\alpha+1)} \phi_q\left(\int_0^1 H(1,\tau)\, y(\tau)\, d\tau\right) \\
& + \frac{\gamma}{1-\gamma} \\
& \times \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(s,\tau)\, y(\tau)\, d\tau\right) ds \\
& + \frac{\lambda+\gamma\mu h}{1-\gamma}
\end{aligned} \tag{56}$$

for $t \in [0,1]$, and

$$\begin{aligned}
u(t) \geqslant & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 s^{\beta-1} H(1,\tau)\, y(\tau)\, d\tau\right) ds \\
& + \frac{\gamma}{1-\gamma} \\
& \times \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(s,\tau)\, y(\tau)\, d\tau\right) ds \\
& + \frac{\lambda+\gamma\mu h}{1-\gamma} \\
= & \int_0^t \alpha(t-s)^{\alpha-2} \phi_q\left(s^{\beta-1}\right) ds \frac{1}{\Gamma(\alpha+1)} \phi_q \\
& \times \left(\int_0^1 H(1,\tau)\, y(\tau)\, d\tau\right) \\
& + \frac{\gamma}{1-\gamma} \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \\
& \qquad \times \left(\int_0^1 H(s,\tau)\, y(\tau)\, d\tau\right) ds + \frac{\lambda+\gamma\mu h}{1-\gamma} \\
\geqslant & c_\delta \frac{1}{\Gamma(\alpha+1)} \phi_q\left(\int_0^1 H(1,\tau)\, y(\tau)\, d\tau\right) \\
& + \frac{\gamma}{1-\gamma} \\
& \times \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(s,\tau)\, y(\tau)\, d\tau\right) ds \\
& + \frac{\lambda+\gamma\mu h}{1-\gamma} \\
\geqslant & c_\delta \left[\frac{1}{\Gamma(\alpha+1)} \phi_q\left(\int_0^1 H(1,\tau)\, y(\tau)\, d\tau\right)\right. \\
& + \frac{\gamma}{1-\gamma} \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \\
& \qquad \times \phi_q\left(\int_0^1 H(s,\tau)\, y(\tau)\, d\tau\right) ds \\
& \left. + \frac{\lambda+\gamma\mu h}{1-\gamma}\right]
\end{aligned} \tag{57}$$

for $t \in [\delta, 1]$. Therefore, $u(t) \geqslant c_\delta \|u\|$ for $\delta \leqslant t \leqslant 1$. The proof is complete. □

Theorem 21. *Assume that (H1), (H4) hold. Then, the fractional differential equation boundary-value problem* (3) *has no positive solution for* $\lambda + \gamma\mu h > (1-\gamma)e$.

Proof. Assume, to the contrary, the fractional differential equation boundary-value problem (3) has a positive solution $u(t)$ for $\lambda + \gamma\mu h > (1-\gamma)e$. Then, by Lemma 8, we have

$$\begin{aligned}
u(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \\
& \times \left(\int_0^1 H(s,\tau)\, a(\tau)\, f(u(\tau))\, d\tau\right) ds \\
& + \frac{\gamma}{1-\gamma} \\
& \times \int_0^h \frac{(h-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\int_0^1 H(s,\tau)\, a(\tau)\, f(u(\tau))\, d\tau\right) ds \\
& + \frac{\lambda+\gamma\mu h}{1-\gamma}.
\end{aligned} \tag{58}$$

Therefore, $u(t) > e$ on $[0,1]$. In view of (32) and (33), we obtain

$$\begin{gathered}
f(u(t)) \geqslant N\phi_p(u(t)) \quad \text{on } [0,1], \\
c_\delta \phi_q(N)\, \phi_q\left(\int_\delta^1 H(1,\tau)\, a(\tau)\, d\tau\right) \\
\times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(s^{\beta-1}\right) ds > 1.
\end{gathered} \tag{59}$$

Then, by Lemmas 6 and 9, we obtain

$$\begin{aligned}
\|u\| \geq & u(1) \\
> & \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \\
& \times \left(\int_0^1 H(s,\tau)\, a(\tau)\, f(u(\tau))\, d\tau\right) ds
\end{aligned}$$

$$\geqslant \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(s^{\beta-1}\right) ds \phi_q$$
$$\times \left(\int_0^1 H(1,\tau) a(\tau) f(u(\tau)) d\tau\right)$$
$$\geqslant \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(s^{\beta-1}\right) ds \phi_q(N) \phi_q$$
$$\times \left(\int_\delta^1 H(1,\tau) a(\tau) \phi_p(u(\tau)) d\tau\right)$$
$$\geqslant \|u\| c_\delta \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(s^{\beta-1}\right) ds \phi_q(N) \phi_q$$
$$\times \left(\int_\delta^1 H(1,\tau) a(\tau) d\tau\right)$$
$$> \|u\|. \tag{60}$$

This contradiction completes the proof. □

Corollary 22. *Assume that (H1) holds and $f_\infty = +\infty$. Then, the fractional differential equation boundary-value problem* (3) *has no positive solution for sufficiently large $\lambda > 0$.*

6. Conclusion: Identities on the Special Polynomials whereby Caputo Fractional Derivative

In this final part, we will focus on the new interesting identities related to special polynomials by means of Caputo fractional derivative.

As well known, the Bernoulli polynomials may be defined to be

$$F(t,z) = \frac{z}{e^z - 1} e^{tz} = e^{Bz} = \sum_{n=0}^{\infty} B_n(t) \frac{z^n}{n!}, \tag{61}$$

where usual convention about replacing B^n by B_n in is used. Also, we note that the Bernoulli polynomials is analytic on the region $D = \{z \in \mathbb{C} \mid |z| < 2\pi\}$ (see [33]).

Let d/dt be familiar normal derivative, then we can obtain the following identity

$$\frac{d}{dt} t^n = n t^{n-1}. \tag{62}$$

Differentiating in both sides of (61), we have

$$\frac{d}{dt} B_n(t) = n B_{n-1}(t) \tag{63}$$

(see [33]).

When $t = 0$ in (61), we have $B_n(0) := B_n$ are called Bernoulli numbers, which can be generated by

$$F(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \tag{64}$$

By (61) and (64), we have the following functional equation:

$$F(t,z) = e^{tz} F(z) \tag{65}$$

and this equation yields to

$$\begin{aligned} B_m(t) &= \sum_{k=0}^{m} \binom{m}{k} t^{m-k} B_k \\ &= \sum_{k=0}^{m} \binom{m}{k} t^k B_{m-k} \end{aligned} \tag{66}$$

(see [33]).

Let us now take $y(t) = B_m(t)$ in Definition 2 leads to

$$\begin{aligned} D_{0+}^{\alpha} B_m(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\left(d^n/dt^n\right) B_m(t)\big|_{t=s}}{(t-s)^{\alpha-n+1}} ds \\ &= m(m-1)\cdots(m-n+1) \\ &\quad \times \sum_{k=0}^{m-n} \binom{m-n}{k} B_{m-n-k} \\ &\quad \times \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{s^k}{(t-s)^{\alpha-n+1}} ds\right] \\ &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)} \sum_{k=0}^{m-n} \frac{k! \binom{m-n}{k} B_{m-n-k}}{\Gamma(n+k-\alpha+1)} t^{k-\alpha+n}. \end{aligned} \tag{67}$$

Therefore, we procure the following theorem.

Theorem 23. *The following identity holds true:*

$$\begin{aligned} D_{0+}^{\alpha} B_m(t) &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)} \\ &\quad \times \sum_{k=0}^{m-n} \frac{k! \binom{m-n}{k} B_{m-n-k}}{\Gamma(n+k-\alpha+1)} t^{k-\alpha+n}. \end{aligned} \tag{68}$$

In [34], the Bernoulli polynomials of higher order are defined by

$$\underbrace{\frac{z}{e^z-1} \frac{z}{e^z-1} \cdots \frac{z}{e^z-1}}_{l\text{-times}} e^{tz} = \sum_{m=0}^{\infty} B_m^{(l)}(t) \frac{z^n}{n!}, \tag{69}$$

we note that $B_m^{(l)}(t)$ is analytic on D. It follows from (69), we have

$$\begin{aligned} \frac{d}{dt} B_m^{(l)}(t) &= m B_{m-1}^{(l)}(t), \\ \frac{d^n}{dt^n} B_m^{(l)}(t) &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)} B_{m-n}^{(l)}(t) \end{aligned} \tag{70}$$

(see [34]).

Substituting $x = 0$ into (69), $B_m^{(l)}(0) := B_m^{(l)}$ are called Bernoulli polynomials of higher order.

Owing to (69) and (70), we readily see that

$$\begin{aligned} D^{\alpha}_{0+}B^{(l)}_m(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{(d^n/dt^n)B^{(l)}_m(t)\big|_{t=s}}{(t-s)^{\alpha-n+1}}ds \\ &= m(m-1)\cdots(m-n+1) \\ &\quad\times\sum_{k=0}^{m-n}\binom{m-n}{k}B^{(l)}_{m-n-k} \\ &\quad\times\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{s^k}{(t-s)^{\alpha-n+1}}ds\right] \\ &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}B^{(l)}_{m-n-k}}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n} \\ &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n} \\ &\quad\times\left(\sum_{\substack{s_1+s_2+\cdots+s_l=m-n-k\\ s_l\geq 0}}\binom{m-n-k}{s_1,s_2,\ldots,s_l}\left(\prod_{j=1}^{l}B_{s_j}\right)\right). \end{aligned} \tag{71}$$

Therefore, we can state the following theorem.

Theorem 24. *The following identity holds true:*

$$\begin{aligned} D^{\alpha}_{0+}B^{(l)}_m(t) &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n} \\ &\quad\times\left(\sum_{\substack{s_1+s_2+\cdots+s_l=m-n-k\\ s_l\geq 0}}\binom{m-n-k}{s_1,s_2,\ldots,s_l}\left(\prod_{j=1}^{l}B_{s_j}\right)\right), \end{aligned} \tag{72}$$

in which B_{s_j} and $\binom{m-n-k}{s_1,s_2,\ldots,s_l}$ are Bernoulli numbers and multinomial coefficients.

In the region $T = \{z \in \mathbb{C} \mid |z| < \pi\}$, the Euler polynomials and the Euler polynomials of higher order are given, respectively, with the help of the following generating functions:

$$\frac{2}{e^z+1}e^{tz} = \sum_{m=0}^{\infty}E_m(t)\frac{z^m}{m!}, \tag{73}$$

$$\begin{aligned} &\underbrace{\frac{2}{e^z+1}\frac{2}{e^z+1}\cdots\frac{2}{e^z+1}}_{l\text{-times}}e^{tz} \\ &\quad= \sum_{m=0}^{\infty}\left(\sum_{s_1+s_2+\cdots+s_l=m}\binom{m}{s_1,s_2,\ldots,s_l}\left(\prod_{j=1}^{l-1}E_{s_j}\right)t^{s_l}\right)\frac{z^m}{m!} \\ &\quad= \sum_{m=0}^{\infty}E^{(l)}_m(t)\frac{z^n}{n!}, \end{aligned} \tag{74}$$

where E_{s_j} are Euler numbers in (see [33–36]). From the last equation, we discover the following:

$$\begin{aligned} \frac{d}{dt}E_m(t) &= mE_{m-1}(t), \\ \frac{d}{dt}E^{(l)}_m(t) &= mE^{(l)}_{m-1}(t) \end{aligned} \tag{75}$$

(see [35]).

Obviously, we have that

$$E^{(1)}_m(t) := E_m(t). \tag{76}$$

Taking $y(t) = E^{(l)}_m(t)$ in Definition 2, by (73) and (75), we compute

$$\begin{aligned} D^{\alpha}_{0+}E^{(l)}_m(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{(d^n/dt^n)E^{(l)}_m(t)\big|_{t=s}}{(t-s)^{\alpha-n+1}}ds \\ &= m(m-1)\cdots(m-n+1) \\ &\quad\times\sum_{k=0}^{m-n}\binom{m-n}{k}E^{(l)}_{m-n-k} \\ &\quad\times\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{s^k}{(t-s)^{\alpha-n+1}}ds\right] \\ &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}E^{(l)}_{m-n-k}}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n} \\ &= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n} \\ &\quad\times\left(\sum_{\substack{s_1+s_2+\cdots+s_l=m-n-k\\ s_l\geq 0}}\binom{m-n-k}{s_1,s_2,\ldots,s_l}\left(\prod_{j=1}^{l}E_{s_j}\right)\right). \end{aligned} \tag{77}$$

Therefore, we obtain the following theorem.

Theorem 25. *The following identity*

$$\begin{aligned} &D^{\alpha}_{0+}E^{(l)}_m(t) \\ &\quad= \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}}{\Gamma(n+k-\alpha+1)} \\ &\quad\quad\times\left(\sum_{\substack{s_1+s_2+\cdots+s_l=m-n-k\\ s_l\geq 0}}\binom{m-n-k}{s_1,s_2,\ldots,s_l}\left(\prod_{j=1}^{l}E_{s_j}\right)\right)t^{k-\alpha+n} \end{aligned} \tag{78}$$

is true. Obviously, we have that

$$D^{\alpha}_{0+}E_m(t) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}E_{m-n-k}}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n}. \tag{79}$$

In the region $T = \{z \in \mathbb{C} \mid |z| < \pi\}$, Genocchi polynomials, $G_m(x)$, and Genocchi polynomials of higher order, $G_m^{(l)}(x)$, are defined as an extension of Genocchi numbers G_m defined in [33, 37, 38], respectively,

$$\frac{2z}{e^z+1}e^{tz} = \sum_{m=0}^{\infty} G_m(t)\frac{z^m}{m!},$$
$$\underbrace{\frac{2z}{e^z+1}\frac{2z}{e^z+1}\cdots\frac{2z}{e^z+1}}_{l\text{-times}}e^{tz} = \sum_{m=0}^{\infty} G_m^{(l)}(t)\frac{z^n}{n!}. \tag{80}$$

In this final section, by the similar method, we arrive at the following theorem.

Theorem 26. *The following identity*

$$D_{0+}^{\alpha}G_m^{(l)}(t) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}}{\Gamma(n+k-\alpha+1)} \times \left(\sum_{\substack{s_1+s_2+\cdots+s_l=m-n-k\\ s_l\geq 0}}\binom{m-n-k}{s_1,s_2,\ldots,s_l}\left(\prod_{j=1}^{l}G_{s_j}\right)\right)t^{k-\alpha+n} \tag{81}$$

is true. Obviously, we have that,

$$D_{0+}^{\alpha}G_m(t) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}G_{m-n-k}}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n}. \tag{82}$$

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