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THE LATTICE OF SUBHYPERGROUPS OF A HYPERGROUP

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ABSTRACT

Tarnauceanu [On the poset of subhypergroups of a hypergroup, Int. J. Open Problems Comp. Math. 3(2) (2010) 505-508] gave some open problems concerning to the set of subhypergroups of a hypergroup, partially ordered by set inclusion. In this study, we obtain that some certain subposets of subhypergroups of a hypergroup are modular or distributive lattice.

Keywords: Hypergroups; Subhypergroups; Invertible; Lattice; Distributive lattice

1. INTRODUCTION

One of the most important problems in algebra is to obtain some of the properties of algebraic structures by examining the lattice of subalgebraic structures (such as subgroups of a group, ideals of a ring, submodules of a module or ideals of a lattice). Until now, many important results have been obtained in this subject, particularly, in group theory [7]. As a generalization of algebraic structures, hyper structure was introduced by Marty [4] in 1934. Since then this theory has enjoyed a rapid development.

In this study, we investigate the properties of closed, invertible, ultraclosed and conjugable subhypergroups classes. We study when the hypergroups satisfy the property that the hyperproduct of subhypergroups becomes an operation on the set of subhypergroups. It is investigated in which cases, the poset of the subhypergroups of a hypergroup is a lattice. It is examined when this lattice is modular or distributive. Thus some information about a hypergroup may be obtained by investigating the lattice of its subhypergroups.

2. PRELIMINARIES

We first give some fundamental definitions and results from literature. For more details, we refer to the references quoted from [1-3].

Now we first introduce the lattice-theoretic base of our work. Let L be a lattice where "$\leq$" denotes the partial ordering of L, the join (sup) and meet (inf) of the elements of L are denoted by "$\vee$" and "$\wedge$", respectively. We also write 1 and 0 for top and bottom elements of L, respectively. We say that L is a complete lattice if L is closed with respect to arbitrary suprema and arbitrary infima. It is well known that a nonempty ordered set L is a complete lattice if it is closed under arbitrary infima.
A lattice L is called distributive if, for any $x, y, z \in L$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice L is called modular if, for any $x, y, z \in L$ with $x \leq y$,

$$x \vee (y \wedge z) = y \wedge (x \vee z).$$

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A lattice is distributive if and only if it is modular and does not contain a sublattice isomorphic to the five element lattice M_5.

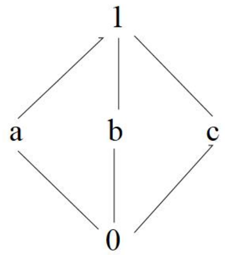

Figure 1. Hasse diagram of the lattice M_5

Let H be a nonempty set and let $P^*(H)$ be the set of all nonempty subsets of H. Then a hyperoperation on H is a map $\circ: H \times H \longrightarrow P^*(H)$ and the couple $(H,\circ)$ is called a hypergroupoid.

For any two nonempty subsets A and B of H and $x \in H$, the sets $A \circ B$, $A \circ x$ and $x \circ A$ are defined by:

$$A \circ B = \bigcup \{a \circ b \mid a \in A, b \in B\},$$

$$A \circ x = A \circ \{x\},$$

$$x \circ A = \{x\} \circ A.$$

The set $A \circ B$ is called hyperproduct of A and B.
A hypergroupoid $(H,\circ)$ is called semihypergroup if for all a, b, c of H we have
$a \circ (b \circ c) = (a \circ b) \circ c$.
A hypergroupoid $(H,\circ)$ is called quasihypergroup if for all a of H we have
$a \circ H = H \circ a = H$.
A hypergroup is a hypergroupoid which is both a semihypergroup and a quasihypergroup.

$(H,\circ)$ is called a commutative hypergroup of H if $x \circ y = y \circ x$ for all $x, y \in H$. A nonempty subset K of a hypergroup $(H,\circ)$ is called a subhypergroup of H if $K \circ K \subseteq K$ and K is a hypergroup under the hyperoperation $\circ$. In other words, it is a hypergroup according to the hyperoperation on H. K provides the following conditions:

1. $a \circ b \subseteq K$ for all $a, b \in K$.
2. $a \circ K = K \circ a = K$, for all $a \in K$.

A nonempty subset K of a semihypergroup $(H,\circ)$ is called a complete part of H if the following implication holds:

For all $n \geq 2$, $n \in \mathbb{N}$ and for all $(x_1, \ldots, x_n) \in H^n$, $\prod_{i=1}^{n} x_i \cap K \neq \emptyset \Longrightarrow \prod_{i=1}^{n} x_i \subset K$.

Definition 2.1 [3] A subhypergroup K of a hypergroup $(H,\circ)$ is called

- closed if for all $a, b \in K$ and $x \in H$, from $a \in x \circ b$ and $a \in b \circ x$, it follows that $x \in K$;
- invertible if $a \in b \circ K$ implies $b \in a \circ K$ and $a \in K \circ b$ implies $b \in K \circ a$ for any $a, b \in H$;
- ultraclosed if for all $x \in H$, we have $K \circ x \cap (H \backslash K) \circ x = \emptyset$ and $x \circ K \cap x \circ (H \backslash K) = \emptyset$;
- conjugable if it is closed and for all $x \in H$, there exists $x', y' \in H$ such that $x' \circ x \subseteq K$ and $x \circ y' \subseteq K$.

Note that the sets of all subhypergroups, closed subhypergroups, invertible subhypergroups, ultraclosed subhypergroups and conjugable subhypergroups of H will be denoted by $Sub(H)$, $CSub(H)$, $ISub(H)$, $USub(H)$ and $ConSub(H)$, respectively. Moreover, we remark that these sets are partially ordered under the set inclusion. It is clear that $K \circ K = K$ for any K subhypergroup of H and $ConSub(H) \subseteq USub(H) \subseteq ISub(H) \subseteq CSub(H) \subseteq Sub(H)$.

In the literarure, the set $\{e \in H \ | \exists x \in H \ such \ that \ x \in x \circ e \cup e \circ x\}$ denoted by I_P and it is called the set of partial identities of H.

Theorem 2.2 [3, Theorem 2.3.14] A subhypergroups K of a hypergroup $(H,\circ)$ is ultraclosed if and only if K is closed and $I_P \subseteq K$.

Theorem 2.3 [6, Corolary 2.11] The non-void intersection of two closed subhypergroups is a closed subhypergroup.

Theorem 2.4 [2, Theorem 36] A subhypergroups K of a hypergroup H is a complete part if and only if K is conjugable.

3. MAIN RESULTS

The following proposition can be easily from the axioms:
Proposition 3.1 Let $(H,\circ)$ be a hypergroup . Then

- if $A \subseteq B$, then $A \circ C \subseteq B \circ C$ and $C \circ A \subseteq C \circ B$ for every $A, B, C \in P^*(H)$,
- if $a \in A$, then $B \circ a \subseteq B \circ A$ and $a \circ B \subseteq A \circ B$ for every $A, B \in P^*(H)$,
- if $a \in A$ and $b \in B$, then $a \circ b \subseteq A \circ B$ for every $A, B \in P^*(H)$.

Lemma 3.2 Let K_1 and K_2 be invertible subhypergroups of H. Then $K_1 \circ K_2 = K_2 \circ K_1$ if and only if $K_1 \circ K_2$ and $K_2 \circ K_1$ are invertible subhypergroups of H.

Proof. Assume that $K_1 \circ K_2 = K_2 \circ K_1$. Let $a \in K_1 \circ K_2$. Then there exist $k_2 \in K_2$ such that $a \in K_1 \circ k_2$. Since K_1 is invertible, $k_2 \in K_1 \circ a$. Hence,

$$\begin{aligned} K_1 \circ K_2 &= K_1 \circ K_2 \circ k_2 \\ &\subseteq K_1 \circ K_2 \circ K_1 \circ a \\ &= K_1 \circ K_1 \circ K_2 \circ a \\ &= K_1 \circ K_2 \circ a \end{aligned}$$

On the other hand, since $a \in K_1 \circ K_2$, there exist $k_1 \in K_1$ and $k_2 \in K_2$ such that $a \in k_1 \circ k_2$.

$$\begin{aligned} K_1 \circ K_2 \circ a &\subseteq K_1 \circ K_2 \circ k_1 \circ k_2 \\ &= K_2 \circ K_1 \circ k_1 \circ k_2 \\ &= K_2 \circ K_1 \circ k_2 \\ &= K_1 \circ K_2 \circ k_2 \\ &= K_1 \circ K_2 \end{aligned}$$

Hence $K_1 \circ K_2 \circ a = K_1 \circ K_2$ for all $a \in K_1 \circ K_2$. Similarly, $a \circ K_1 \circ K_2 = K_1 \circ K_2$.
Let $a, b \in K_1 \circ K_2$.

$$\begin{aligned} a \circ b &\subseteq K_1 \circ K_2 \circ K_1 \circ K_2 \\ &= K_1 \circ K_1 \circ K_2 \circ K_2 \\ &= K_1 \circ K_2 \end{aligned}$$

Hence $K_1 \circ K_2 \in Sub(H)$. Let $x \in K_1 \circ K_2 \circ y$. Then there exists a $k \in K_2 \circ y$ such that $x \in K_1 \circ k$. Since K_1 and K_2 are invertible subhypergroups, we obtain that $y \in K_2 \circ k$ and $k \in K_1 \circ x$. Thus $y \in$

$K_2 \circ K_1 \circ x = K_1 \circ K_2 \circ x$. Similarly, it is easily seen that if $x \in y \circ K_1 \circ K_2$, then $y \in x \circ K_1 \circ K_2$. Hence $K_1 \circ K_2$ is invertible subhypergroup of H.

Conversely, assume that $K_1 \circ K_2$ and $K_2 \circ K_1$ are invertible subhypergroups of H. Firstly we prove that $K_2 \circ K_1 \circ K_2 \subseteq K_1 \circ K_2$. Let $a \in K_2 \circ K_1 \circ K_2$. Since $K_1 \circ K_2 \circ K_1 \circ K_2 = K_1 \circ K_2$, it follows that $K_1 \circ a \subseteq K_1 \circ K_2$. Then, since $K_1 \circ a \neq \emptyset$, we have there exists $x \in K_1 \circ K_2$ such that $x \in K_1 \circ a$. Since K_1 is invertible, $a \in K_1 \circ x \subseteq K_1 \circ K_1 \circ K_2 = K_1 \circ K_2$. Hence $K_2 \circ K_1 \circ K_2 \subseteq K_1 \circ K_2$.

Now, we show that $K_2 \circ K_1 \subseteq K_1 \circ K_2$. Let $b \in K_2 \circ K_1$. Then $b \circ K_2 \subseteq K_2 \circ K_1 \circ K_2$. Hence $b \circ K_2 \subseteq K_1 \circ K_2$. Here, we note that $b \circ K_2 \neq \emptyset$, then there exists $x \in K_1 \circ K_2$ such that $x \in b \circ K_2$. Since K_2 is invertible, $b \in x \circ K_2 \subseteq K_1 \circ K_2 \circ K_2 = K_1 \circ K_2$. Hence $K_2 \circ K_1 \subseteq K_1 \circ K_2$. Similarly, $K_1 \circ K_2 \subseteq K_2 \circ K_1$. Consequently, $K_1 \circ K_2 = K_2 \circ K_1$.

The following example shows that the hyperproduct of two closed subhypergroups of a hypergroup may not be closed subhypergroup.

Example 3.3 Define the following hyperoperation on the real set $\mathbb{R}$:

$$x \circ y = \begin{cases} \{x\}, & if \quad x = y, \\ (min\{x,y\}, max\{x,y\}) & if \quad x \neq y. \end{cases}$$

Then $(\mathbb{R},\circ)$ is a hypergroup [3].
It is easy to see that $\{1\}$ and $\{2\}$ are closed subhypergroups of $(\mathbb{R},\circ)$. The hyperproduct of $\{1\}$ and $\{2\}$ $\{1\} \circ \{2\} = (1,2)$ is a subhypergroup, but it is not closed.

Lemma 3.4 Let $(H,\circ)$ be a hypergroup such that $a \circ b \cap \{a, b\} \neq \emptyset$ for all $a, b \in H$. If $K_1, K_2 \in CSub(H)$ or $K_1, K_2 \in ISub(H)$, then $K_1 \cap K_2 \neq \emptyset$.

Proof. Suppose that $K_1, K_2 \in CSub(H)$. Let $a \in K_1$ and $b \in K_2$. Since $(a \circ b) \cap \{a, b\} \neq \emptyset$, $a \in a \circ b$ or $b \in a \circ b$. Since K_1, K_2 are closed subhypergroups, $b \in K_1$ or $a \in K_2$. Therefore, $a \in K_1 \cap K_2$ or $b \in K_1 \cap K_2$. Hence $K_1 \cap K_2 \neq \emptyset$. If $K_1, K_2 \in ISub(H)$, then, with the above explanations, $a \in K_1 \circ b$ or $b \in a \circ K_2$. This implies $b \in K_1 \circ a = K_1$ or $a \in b \circ K_2 = K_2$ and hence $K_1 \cap K_2 \neq \emptyset$.

Theorem 3.5 Let $(H,\circ)$ be a commutative hypergroup such that $a \circ b \cap \{a, b\} \neq \emptyset$ for all $a, b \in H$. Then $(ISub(H), \subseteq)$ is a lattice such that $K_1 \vee K_2 = K_1 \circ K_2$ and $K_1 \wedge K_2 = K_1 \cap K_2$ for all $K_1, K_2 \in ISub(H)$.

Proof. Let $K_1, K_2 \in ISub(H)$. Then by Lemma 3.2, we have $K_1 \circ K_2$ is an invertible subhypergroup. Firstly we prove that $K_1 \cup K_2 \subseteq K_1 \circ K_2$. Let $x \in K_1$. Then, since $K_2 \neq \emptyset$, there exists $b \in K_2$. Therefore, by hypothesis, $x \in x \circ b$ or $b \in x \circ b$. If $x \in x \circ b$, then $x \in K_1 \circ K_2$. If $b \in x \circ b$, then $b \in x \circ K_2$. Thus $x \in b \circ K_2 = K_2$. Therefore $x \in x \circ x \subseteq K_1 \circ K_2$. Hence $K_1 \subseteq K_1 \circ K_2$. Similarly, $K_2 \subseteq K_1 \circ K_2$. Hence, $K_1 \cup K_2 \subseteq K_1 \circ K_2$.
Now we prove that $K_1 \circ K_2$ is the smallest invertible subhypergroup containing K_1 and K_2. Let K_3 be a invertible subhypergroup such that $K_1 \subseteq K_3$ and $K_2 \subseteq K_3$. Then $K_1 \circ K_2 \subseteq K_3 \circ K_3 = K_3$. Consequently $K_1 \vee K_2 = K_1 \circ K_2$.
Next, we prove that $K_1 \wedge K_2 = K_1 \cap K_2$. By Lemma 3.4, $K_1 \cap K_2 \neq \emptyset$. It is enough to show that $K_1 \cap K_2 \in ISub(H)$. It is clear that $a \circ b \subseteq K_1 \cap K_2$ and $a \circ (K_1 \cap K_2) \subseteq (K_1 \cap K_2)$ for all $a, b \in K_1 \cap K_2$. Let $x \in K_1 \cap K_2$. As $x \in a \circ K_1$ and $x \in a \circ K_2$, there exist $k_1 \in K_1, k_2 \in K_2$ such that $x \in a \circ k_1, x \in a \circ k_2$. Inside the proof of Lemma 3.4, we have $k_1 \in K_1 \cap K_2$ or $k_2 \in K_1 \cap K_2$. Therefore, $x \in a \circ (K_1 \cap K_2)$. This implies that $K_1 \cap K_2 = a \circ (K_1 \cap K_2)$ for every $a \in K_1 \cap K_2$. By the commutativity, $K_1 \cap K_2 = (K_1 \cap K_2) \circ a$. Hence, $K_1 \cap K_2 \in Sub(H)$. Further, if $c \in d \circ (K_1 \cap K_2)$, then $c \in d \circ K_1$ and $c \in d \circ K_2$. As K_1, K_2 are invertible, $d \in c \circ K_1$ and $d \in c \circ K_2$. By repeating the above techniques, $d \in c \circ (K_1 \cap K_2)$. By the commutativity, $K_1 \cap K_2 \in ISub(H)$.

Theorem 3.6 Let $(H,\circ)$ be a commutative hypergroup such that $a \circ b \cap \{a, b\} \neq \emptyset$ for all $a, b \in H$. Then $(ISub(H), \subseteq)$ is a distributive lattice.

Proof. Let $K_1, K_2, K_3 \in ISub(H)$. Since the distributive inequality is valid for every lattice, we have $(K_1 \wedge K_2) \vee (K_1 \wedge K_3) \subseteq K_1 \wedge (K_2 \vee K_3)$.
Let $x \in K_1 \wedge (K_2 \vee K_3)$. Then $x \in K_1 \cap (K_2 \circ K_3)$. Since $x \in K_2 \circ K_3$, there exists $b \in K_2$ and $c \in K_3$ such that $x \in b \circ c$. We know that $b \in b \circ c$ or $c \in b \circ c$. Since K_2 and K_3 are invertible subhypergroups of H, we obtain $b \in K_3$ or $c \in K_2$.
If $c \in K_2$, then $x \in b \circ c \subseteq K_2$ and so $x \in K_1 \cap K_2$. Similarly, if $b \in K_3$, then $x \in K_1 \cap K_3$. Hence $K_1 \cap (K_2 \circ K_3) \subseteq (K_1 \cap K_2) \cup (K_1 \cap K_3)$. By Theorem 3.5, $(K_1 \cap K_2) \cup (K_1 \cap K_3) \subseteq (K_1 \cap K_2) \circ (K_1 \cap K_3)$. Hence, $K_1 \cap (K_2 \circ K_3) \subseteq (K_1 \cap K_2) \circ (K_1 \cap K_3)$. It follows that $(K_1 \wedge K_2) \vee (K_1 \wedge K_3) = K_1 \wedge (K_2 \vee K_3)$. This establishes the result.

Even if a hypergroup satisfies the condition of Theorem 3.6, $(ISub(H), \subseteq)$ is not be a complete lattice. We can see this in the following example.

Example 3.7 Let us consider the group $(\mathbb{Z}, +)$ and the subgroups $S_i = 2^i\mathbb{Z}$, where $i \in \mathbb{N}$. For any $x \in \mathbb{Z}\backslash\{0\}$, there exists a unique integer $n(x)$, such that $x \in S_{n(x)} \backslash S_{n(x)+1}$. Define the following commutative hyperoperation on $\mathbb{Z}\backslash\{0\}$:
if $n(x) < n(y)$, then $x \circ y = x + S_{n(y)}$;
if $n(x) = n(y)$, then $x \circ y = S_{n(x)} \backslash \{0\}$;
if $n(x) > n(y)$, then $x \circ y = y + S_{n(x)}$.
Notice that if $n(x) < n(y)$, then $n(x + y) = n(x)$. Then $(\mathbb{Z}\backslash\{0\},\circ)$ is a commutative hypergroup and for all $i \in \mathbb{N}$, $S_i\backslash\{0\}$ is an invertible subhypergroup of $\mathbb{Z}\backslash\{0\}$ [3]. Also $ISub(\mathbb{Z}\backslash\{0\}) = \{S_i\backslash\{0\} |\ i \in \mathbb{N}\}$. Hence we obtain that $ISub(\mathbb{Z}\backslash\{0\})$ is distributive lattice.
$\bigcap_{i \in \mathbb{N}} S_i\backslash\{0\} = \emptyset$. So it isn't complete lattice, although $ISub(\mathbb{Z}\backslash\{0\})$ is lattice.

Lemma 3.8 Let $(H,\circ)$ be a hypergroup. If $K_1, K_2 \in USub(H)$, then $K_1 \cap K_2 \in USub(H)$.

Proof. Let $K_1, K_2 \in USub(H)$. K_1, K_2 are closed subhypergroup. Then, by Theorem 2.2, $I_P \subseteq K_1$ and $I_P \subseteq K_2$. Using $I_P \neq \emptyset$, we obtain that $K_1 \cap K_2 \neq \emptyset$. Hence $K_1 \cap K_2 \in CSub(H)$. According to Theorem 2.3, $K_1 \cap K_2 \in USub(H)$, since $I_P \subseteq K_1 \cap K_2$.

Lemma 3.9 Let $(H,\circ)$ be a hypergroup and $K_1, K_2 \in USub(H)$. $K_1 \circ K_2 = K_2 \circ K_1$ if and only if $K_1 \circ K_2$ and $K_2 \circ K_1$ are ultraclosed subhypergroups of H.

Proof. Let $K_1, K_2 \in USub(H)$ such that $K_1 \circ K_2 = K_2 \circ K_1$. According to Lemma 3.2, $K_1 \circ K_2, K_2 \circ K_1 \in ISub(H)$.
It is easily seen that $I_P \subseteq I_P \circ I_P$ and $I_P \subseteq K_1 \circ K_2$. Using Theorem 2.2, we obtain that $K_1 \circ K_2 \in USub(H)$.
Conversely, let $K_1 \circ K_2$ and $K_2 \circ K_1$ be ultraclosed subhypergroups of H. By Lemma 3.2, $K_1 \circ K_2 = K_2 \circ K_1$.

Theorem 3.10 Let $(H,\circ)$ be a commutative hypergroup. Then $(USub(H), \subseteq)$ is a lattice such that $K_1 \vee K_2 = K_1 \circ K_2$ and $K_1 \wedge K_2 = K_1 \cap K_2$ for all $K_1, K_2 \in USub(H)$.

Proof. Let K_1 and K_2 be ultraclosed. Since $K_1 \subseteq K_1 \circ I_P \subseteq K_1 \circ K_2$, we obtain that $K_1 \cup K_2 \subseteq K_1 \circ K_2$. Suppose that $L \in USub(H)$ such that $K_1 \cup K_2 \subseteq L$. Since $K_1 \circ K_2 \subseteq L \circ L = L$, $K_1 \vee K_2 = K_1 \circ K_2$.
By Lemma 3.8, $K_1 \wedge K_2 = K_1 \cap K_2$ for all $K_1, K_2 \in USub(H)$. Hence $(USub(H), \subseteq)$ is a lattice.

Theorem 3.11 Let $(H,\circ)$ be a commutative hypergroup. Then $(USub(H), \subseteq)$ is modular lattice.

Proof. Let K_1, K_2 and K_3 be ultraclosed subhypergroups such that $K_1 \supseteq K_2$.Since the modularity inequality is valid for every lattice, we have $K_2 \vee (K_1 \wedge K_3) \subseteq K_1 \wedge (K_2 \vee K_3)$.
Let $x \in K_1 \wedge (K_2 \vee K_3)$. $x \in K_1$ and $x \in K_2 \circ K_3$. There exists $b \in K_2$ and $c \in K_3$ such that $x \in b \circ c$. We know $x, b \in K_1$.Thus $c \in K_1$ since the closeness of K_1. $x \in b \circ c \subseteq K_2 \circ (K_1 \cap K_3) = K_2 \vee (K_1 \wedge K_3)$.

Hence $K_1 \wedge (K_2 \vee K_3) \subseteq K_2 \vee (K_1 \wedge K_3)$. This establishes that the lattice of ultraclosed subhypergroups of H is a modular lattice.

The following example shows that the lattice $(USub(H), \subseteq)$ does not have to be distributive.
Example 3.12 Let us consider the Klein four-group.

$V =< a, b \ |a^2 = b^2 = (ab)^2 = 1 >$

Then $(V,\circ)$ is a hypergroup with the hyperoperation $a \circ b = \{ab\}$ for all $a, b \in V$. In this situation, $USub(V) = L(V)$, where $L(V)$ is the set of subgroups of V. Since the lattice structure of $L(V)$ forms M_5, $USub(V)$ isn't a distributive lattice.

Lemma 3.13 Let $(H,\circ)$ be a hypergroup and $K_1, K_2 \in ConSub(H)$. The following statements are hold.

1. $K_1 \cap K_2 \in ConSub(H)$
2. $K_1 \circ K_2 = K_2 \circ K_1$ if and only if $K_1 \circ K_2$ and $K_2 \circ K_1$ are conjugable subhypergroups of H.

Proof. Let $K_1, K_2 \in ConSub(H)$.

1. Since Lemma 3.8, $K_1 \cap K_2 \neq \emptyset$. By Theorem 2.4, K_1, K_2 are complete part of H. Now, we show that $K_1 \cap K_2$ is complete part.

Let $(x_1, \dots, x_n) \in H^n$ and $\prod_{i=1}^n x_i \cap (K_1 \cap K_2) \neq \emptyset$. Then

$$\prod_{i=1}^n x_i \cap K_1 \neq \emptyset \quad and \quad \prod_{i=1}^n x_i \cap K_2 \neq \emptyset$$

Since K_1, K_2 are complete part,

$$\prod_{i=1}^n x_i \subset K_1 \quad and \quad \prod_{i=1}^n x_i \subset K_2$$

$$\prod_{i=1}^n x_i \subset K_1 \cap K_2$$

Thus $K_1 \cap K_2$ is complete part. Using Theorem 2.4, $K_1 \cap K_2$ is conjugable subhypergroup of H.

2. Let $x \in H$. Since $K_1, K_2 \in ConSub(H)$, there exists $y_1, y_2 \in H$ such that $x \circ y_1 \subseteq K_1$ and $x \circ y_2 \subseteq K_2$. In this case, $x \circ a \subseteq x \circ y_1 \circ x \circ y_2 \subseteq K_1 \circ K_2$, for all $a \in y_1 \circ x \circ y_2$. So $K_1 \circ K_2$is conjugable subhypergroups of H. Similarly, it is easily seen that $K_2 \circ K_1$ is conjugable. The opposite can be clearly seen with Lemma 3.2 since $ConSub(H) \subseteq ISub(H)$.

Corollary 3.14 Let $(H,\circ)$ be a commutative hypergroup. $(ConSub(H), \subseteq)$ is a lattice. Moreover it is modular.
Proof. It is obvious from Theorem 3.10 and Theorem 3.11.

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REFERENCES

[1] Birhoff G, Lattice Theory, Amer. Math. Soc. Colloq. Publ., 1967.

[2] Corsini P and Leoreanu V, Application of Hyperstructure Theory, Kluwer Academic Publishers, 2003.

[3] Davvaz B and LeoreanuV, Hyperring theory and applications, International Academic Press, 2007.

[4] Marty F. Sur ungeneralisation de la notion degroup, 8th Congress of Scandinavian Mathematicians, 45-49, 1934.

[5] Tarnauceanu M On the poset of subhypergroups of a hypergroup, Int. J. Open Problems Comp. Math. 3(2), 115-122, 2010.

[6] Massouros CG, Some properties of certain subhypergroups, Ratio Mathematica, 25, 67-76, 2013.

[7] Schmidt R, Subgroup Lattices of Groups, de Gruyter Expositions in Mathematics 14, de Gruyter, Berlin, 1994.