On surfaces endowed with canonical principal direction in Euclidean spaces

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Abstract

In this paper, we introduce canonical principal direction (CPD) submanifolds with higher codimension in Euclidean spaces. We obtain the complete classification of surfaces endowed with CPD in the Euclidean 4-space.

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Keywords.

1 Introduction

Let N be a Riemannian manifold, M an immersed hypersurface of N and X a vector field in N. M is said to have canonical principle direction (CPD) with relative to X if the projection of X onto the tangent space of M gives one of principle directions of M, [9]. One of the most common examples of hypersurfaces with CPD is rotational hypersurfaces in Euclidean spaces which have canonical principal direction relative X if X is chosen to be a vector field parallel to its rotation axis.

On the other hand, a submanifold in the Euclidean space is said to be a constant angle surface if there is a constant direction k which makes constant angle with the tangent plane at every point of that surface. There are many classification results for such hypersurfaces called as constant angle (CA) hypersurfaces obtained so far, in different ambient spaces, [1, 6, 8, 10, 11, 13, 15]. Before we proceed, we would like to note that a CAS surface in the Euclidean 3-space has CPD relative to k. Because of this reason, hypersurfaces with CPD relative to a fixed direction in Euclidean spaces have caught interest of some geometers in the recent years. For example, surfaces with CPD in the Euclidean 3-space $\mathbb{E}^3$ have been studied in [16]. Then, this study was moved into the Minkowski 3-space $\mathbb{E}^3_1$ in [12, 17]. Furthermore, CPD surfaces in product spaces also take attention of some geometers. For example, some classification results on surfaces with CPD relative to ∂_t in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ have been obtained in

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[4, 7] (See also [8]), where ∂_t denotes the unit vector field tangent to the second factor.

On the other hand, Tojeiro studied CPD hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ in [18]. Later, Mendonça and Tojeiro give generalization of the notion of CPD hypersurfaces into higher codimensional submanifolds. For this porpuse, they give the definition class $\mathcal{A}$ in [14]. An immersion $f : M^n \to Q_c^n \times \mathbb{R}$ is said to belongs to class $\mathcal{A}$ immersions if the tangential part of ∂_t is one of principal directions of all shape operators of f. By a similar way, we would like to give the following definition of CPD submanifolds in Euclidean spaces.

Definition 1.1. *Let M^n be a submanifold in $\mathbb{E}^m$ and k be a fixed direction in $\mathbb{E}^m$. M is said to be a submanifold endowed with canonical principal direction, (shortly, a CPD submanifold) if the tangential component k^T of k is one of principal directions of all shape operators of M.*

The aim of this paper is to obtain complete classification of CPD surfaces in the Euclidean 4-space $\mathbb{E}^4$. In Sect. 2, we introduce the notation that we will use and give a brief summary of basic definitions in theory of submanifolds in Euclidean spaces. In Sect. 3, we obtain the complete classification of CPD surfaces in the Euclidean 4-space.

2 Prelimineries

Let $\mathbb{E}^m$ denote the Euclidean m-space with the canonical Euclidean metric tensor given by

$$\widetilde{g} = \langle \ , \ \rangle = \sum_{i=1}^{m} dx_i^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$.

Consider an n-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote Levi-Civita connections of $\mathbb{E}^m$ and M by $\widetilde{\nabla}$ and ∇, respectively. The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$

$$\widetilde{\nabla}_X \xi = -S_\xi(X) + D_X \xi, \tag{2}$$

whenever X, Y are tangent and ξ is normal vector field on M, where h, D and S are the second fundamental form, the normal connection and the shape operator of M, respectively. It is well-known that the shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi \rangle = \langle S_\xi X, Y \rangle.$$

The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X, Y)Z, W \rangle = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \tag{3}$$

$$\langle R^D(X, Y)\xi, \eta \rangle = \langle [S_\xi, S_\eta] X, Y \rangle, \tag{4}$$

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \tag{5}$$

whenever X, Y, Z, W are tangent to M, where R, R^D are the curvature tensors associated with connections ∇ and D, respectively. We note that $\bar{\nabla} h$ is defined by

$$(\nabla_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

A submanifold M is said to have flat normal bundle if $R^D = 0$ identically.

The mean curvature vector field H of the surface M is defined as

$$H = \frac{1}{2} trh. \tag{6}$$

If M is a surface, i.e, $n = 2$, then the Gaussian curvature K of the surface M^2 is defined as

$$K = \frac{R(X, Y, X, Y)}{Q(X, Y)}, \tag{7}$$

if X and Y are chosen so that $Q(X, Y) = \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2$ does not vanish.

3   CPD Surfaces in $\mathbb{E}^4$

In this section, we obtain classification of CPD surfaces in $\mathbb{E}^4$.

Let M be a surface in $\mathbb{E}^4$ with CPD relative to k. Without loss of generality, we assume that $k = (1, 0, 0, 0)$. Then, one can define a tangent vector field e_1 and a normal vector field e_3 with the equation

$$k = \cos\theta e_1 + \sin\theta e_3 \tag{8}$$

for a smooth function θ. Let e_2 and e_4 be a unit tangent vector field and a unit normal vector field, satisfying $\langle e_1, e_2 \rangle = 0$ and $\langle e_3, e_4 \rangle = 0$, respectively. By a simple computation considering (8) we obtain the following lemma. Note that we put $h_{ij}^\beta = \langle h(e_i, e_j), e_\beta \rangle = \langle S_\beta e_i, e_j \rangle$, where $S_\beta = S_{e_\beta}$.

Lemma 3.1. *The Levi-Civita connection ∇ of M is given by*

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \tag{9a}$$

$$\nabla_{e_2} e_1 = \tan\theta h_{22}^3 e_2, \qquad \nabla_{e_2} e_2 = -\tan\theta h_{22}^3 e_1. \tag{9b}$$

and the matrix representations of shape operator S of M with respect to $\{e_1, e_2\}$ is

$$S_3 = \begin{pmatrix} -e_1(\theta) & 0 \\ 0 & h_{22}^3 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & h_{22}^4 \end{pmatrix} \tag{10}$$

for functions h_{11}^4, h_{12}^4, h_{22}^3 and h_{22}^4 satisfying

$$e_1(h_{22}^3) = \tan\theta h_{22}^3(h_{11}^3 - h_{22}^3), \tag{11a}$$

$$e_1(h_{22}^4) = -\tan\theta h_{22}^3 h_{22}^4, \tag{11b}$$

$$h_{11}^4 = 0, \quad h_{12}^4 = 0. \tag{11c}$$

Furthermore, θ satisfies

$$e_2(\theta) = 0. \tag{12}$$

Proof. By considering (8) and the normal vector field e_3 being parallel, one can get

$$0 = X(\cos\theta)e_1 + \cos\theta\nabla_X e_1 + \cos\theta h(e_1, X) - \sin\theta S_3 X + X(\sin\theta)e_3 \tag{13}$$

whenever X is tangent to M. (13) for $X = e_1$ gives

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0,$$
$$h_{11}^3 = -e_1(\theta), \tag{14}$$
$$h_{11}^4 = 0.$$

while (13) for $X = e_2$ is giving

$$\nabla_{e_2} e_1 = \tan\theta h_{22}^3 e_2, \quad \nabla_{e_2} e_2 = -\tan\theta h_{22}^3 e_1,$$
$$h_{12}^4 = 0, \quad e_2(\theta) = 0.$$

where e_2 is the other principal direction of M corresponding with the principal curvature h_{22}^3. Thus, we have (9) and (11c) and (12) and the second fundamental form of M becomes

$$h(e_1, e_1) = -e_1(\theta)e_3, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = h_{22}^3 e_3 + h_{22}^4 e_4.$$

By considering the Codazzi equation (5), we obtain (11a) and (11b). $\qquad\square$

Because of (14), if $e_1(\theta) \equiv 0$ implies $h_{11}^3 = 0$. We will consider this particular case seperately.

First assume that $e_1(\theta) \neq 0$. Let p be a a point in M at which $e_1(\theta)$ does not vanish. First, we would like to prove the following lemma.

Lemma 3.2. *There exists a local coordinate system (s, t) defined in a neighborhood $\mathcal{N}_p$ of p such that the induced metric of M is*

$$g = ds^2 + m^2 dt^2 \tag{15}$$

for a smooth function m satisfying

$$e_1(m) - \tan\theta h_{22}^3 m = 0. \tag{16}$$

Furthermore, the vector fields e_1, e_2 described above become $e_1 = \partial_s$, $e_2 = \dfrac{1}{m}\partial_t$ in $\mathcal{N}_p$.

Proof. We have $[e_1, e_2] = -\tan\theta h_{22}^3 e_2$ because of (9). Thus, if m is a non-vanishing smooth function on M satisfying (16), then we have $[e_1, me_2] = 0$. Therefore, there exists a local coordinate system (s, t) such that $e_1 = \partial_s$ and $e_2 = \dfrac{1}{m}\partial_t$. Thus, the induced metric of M is as given in (15). $\qquad\square$

Now, we are ready to obtain the classification theorem.

Theorem 3.3. *Let M be a regular surface in $\mathbb{E}^4$. Let M be a surface endowed with a canonical principal direction relative to $k = (1, 0, 0, 0)$ and assume that the function θ defined in (8) is not constant. Then, M is congruent to the surface given by one of the followings*

1. *A surface given by*

$$x(s, t) = \left(\int_{s_0}^s \cos\theta(\tau)d\tau, \phi_j(t) \int_{s_0}^s \sin\theta(\tau)d\tau \right) + \gamma(t), \quad j = 2, 3, 4 \tag{17a}$$

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where γ is the $\mathbb{E}^4$-valued function given by

$$\gamma(t) = \left(0, \int_{t_0}^t \Psi(\tau)\phi_j{}'(\tau)d\tau\right). \tag{17b}$$

for a function $\Psi \in C^\infty(M)$ and $\phi = \phi(t)$ is the unit speed curve lying on $\mathbb{S}^3(1)$ in $\mathbb{E}^4$;

2. A flat surface given by

$$x(s,t) = \left(\int_{s_0}^s \cos\theta(\tau)d\tau, \phi_j(t_0)\int_{s_0}^s \sin\theta(\tau)d\tau\right) + t_0\phi(t). \tag{18}$$

Here $\phi(t_0)$ and $\phi(t)$ are a constant vector and the unit speed curve lying on $\mathbb{S}^3(1)$ in $\mathbb{E}^4$, respectively.

Conversely, surfaces described above are CPD relative to $k = (1,0,0,0)$.

Proof. In order to proof the necessary condition, we assume that M is a surface endowed with a CPD relative to $k = (1,0,0,0)$ with the isometric immersion $x : M \to \mathbb{E}^4$. Let $\{e_1, e_2; e_3, e_4\}$ be the local orthonormal frame field described as before in Lemma 3.1, h_{11}^3, h_{22}^3 and h_{22}^4 be the principal curvatures of M and (s,t) a local coordinate system given in Lemma 3.2.

Note that, (11a), (11b) and (16) become, respectively

$$(h_{22}^3)_s = -\tan\theta h_{22}^3(\theta' + h_{22}^3), \tag{19}$$

$$(h_{22}^4)_s = -\tan\theta h_{22}^3 h_{22}^4, \tag{20}$$

$$m_s - m\tan\theta h_{22}^3 = 0, \tag{21}$$

Moreover, we have

$$e_1 = x_s. \tag{22}$$

By combining (21) and (20) with (10) we obtain the shape operator S of M as

$$S_3 = \begin{pmatrix} -\theta' & 0 \\ 0 & \cot\theta\frac{m_s}{m} \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{pmatrix} \tag{23}$$

where $'$ denotes ordinary differentiation with respect to the appropriated variable.

By combining (21) and (19) we obtain

$$m_{ss} - \theta'\cot\theta m_s = 0$$

whose general solution is

$$m(s,t) = \Psi_1(t)\int_{s_0}^s \sin\theta(\tau)d\tau + \Psi_2(t)$$

for some smooth functions Ψ_1, Ψ_2. Therefore, by re-defining t properly, we may assume either

$$m(s,t) = \int_{s_0}^s \sin\theta(\tau)d\tau + \Psi(t), \Psi \in C^\infty(M), \tag{24a}$$

or

$$m(s, t) = m(t). \tag{24b}$$

Case 1. Let m satisfies (24a). In this case, by considering the equation (9) with (22), we get the Levi-Civita connection of M satisfies

$$\nabla_{\partial_s}\partial_s = 0, \quad \nabla_{\partial_s}\partial_t = \nabla_{\partial_t}\partial_s = \frac{m_s}{m}\partial_t, \quad \nabla_{\partial_t}\partial_t = -mm_s\partial_s + \frac{m_t}{m}\partial_t.$$

By combining the first equation given above with (23) and using Gauss formula (1), we have

$$x_{ss} = -\theta' e_3. \tag{25}$$

On the other hand, we have $\langle x_s, k \rangle = \cos\theta$ and $\langle x_t, k \rangle = 0$ from the decomposition (8). By considering these equations, we see that x has the form of

$$x(s, t) = \left(\int_{s_0}^{s} \cos\theta(\tau)d\tau, x_2(s, t), x_3(s, t), x_4(s, t) \right) + \gamma(t) \tag{26}$$

for a $\mathbb{E}^4$-valued smooth function $\gamma = (0, \gamma_2, \gamma_3, \gamma_4)$. On the other hand, by considering (22) and (25) in (8), we yield

$$(1, 0, 0, 0) = \cos\theta x_s - \frac{\sin\theta}{\theta'} x_{ss}. \tag{27}$$

By solving (27) and considering $\langle x_s, x_s \rangle = 1$, we obtain

$$x(s, t) = \int_{s_0}^{s} \cos\theta(\tau)d\tau \Big(1, 0, 0, 0 \Big) + \phi(t) \int_{s_0}^{s} \sin\theta(\tau)d\tau + \gamma(t), \tag{28}$$

where $\phi(t) = \Big(0, \phi_2(t), \phi_3(t), \phi_4(t) \Big)$ is the curve lying on $\mathbb{S}^3(1)$ in $\mathbb{E}^4$. Now, by considering $x_{st} = \frac{m_s}{m} x_t$ in (28), we can rewrite this parametrization as

$$x(s, t) = \int_{s_0}^{s} \cos\theta(\tau)d\tau \Big(1, 0, 0, 0 \Big) + \phi(t) \int_{s_0}^{s} \sin\theta(\tau)d\tau + \int_{t_0}^{t} \Psi(\tau)\phi'(\tau)d\tau, \tag{29}$$

where $\Psi = \Psi(t)$ is a smooth function and $'$ denotes ordinary differentiation with respect to the parameter t. Also, since $\langle x_t, x_t \rangle = m^2$, we yield the curve ϕ parameterized by arc-lenght parameter t. Thus, we have the Case (1) of the theorem.

Case 2. Let m satisfy (24b). Here, we can take $m(t) = 1$ by re-defining t properly. In this case, the induced metric given in (15) of M becomes $g = ds^2 + dt^2$, the Levi Civita connection of M satisfies

$$\nabla_{\partial_s}\partial_s = 0, \quad \nabla_{\partial_s}\partial_t = 0, \quad \nabla_{\partial_t}\partial_t = 0. \tag{30}$$

Also, considering $m = 1$ in (11b) and (21), thus (10) becomes

$$S_3 = \begin{pmatrix} -\theta' & 0 \\ 0 & 0 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{31}$$

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Therefore, x and the normal vectors e_3, e_4 satisfy

$$x_{ss} = -\theta' e_3, \quad x_{st} = 0, \qquad x_{tt} = e_4.$$
$$(e_3)_s = -\theta' x_s, \quad (e_3)_t = 0,$$
$$(e_4)_s = 0, \quad (e_4)_t = -x_t.$$

A straightforward computation yields that M is congruent to the surface given in Case (2) of the theorem. Hence, the proof for the necessary condition is obtained.

The proof of sufficient condition follows from a direct computation. $\square$

Now, assume that the function θ defined in (8) satisfied $e_1(\theta) = 0$. In this case, Lemma 3.1 gives

Lemma 3.4. *The Levi-Civita connection ∇ of M is given by*

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \tag{32a}$$
$$\nabla_{e_2} e_1 = \tan\theta h_{22}^3 e_2, \qquad \nabla_{e_2} e_2 = -\tan\theta h_{22}^3 e_1. \tag{32b}$$

and the matrix representations of shape operator S of M with respect to $\{e_1, e_2\}$ is

$$S_3 = \begin{pmatrix} 0 & 0 \\ 0 & h_{22}^3 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & h_{22}^4 \end{pmatrix} \tag{33}$$

and coefficients of the second fundamental form satisfying

$$e_1(h_{22}^3) = -\tan\theta (h_{22}^3)^2, \tag{34a}$$
$$e_1(h_{22}^4) = -\tan\theta h_{22}^3 h_{22}^4, \tag{34b}$$
$$h_{11}^3 = 0, \quad h_{11}^4 = 0, \quad h_{12}^3 = 0, \quad h_{12}^4 = 0. \tag{34c}$$

Note that here the angle θ is a non-zero constant.

Next, we obtain the following local coordinate system on a neighborhood of a point $p \in M$.

Lemma 3.5. *There exists a local coordinate system (s, t) defined in a neighborhood $\mathcal{N}_p$ of p such that the induced metric of M is*

$$g = ds^2 + m^2 dt^2 \tag{35}$$

for a smooth function m satisfying

$$e_1(m) - m\tan\theta h_{22}^3 = 0. \tag{36}$$

Here, the angle θ is a non-zero constant. Furthermore, the vector fields e_1, e_2 described above become $e_1 = \partial_s$, $e_2 = \dfrac{1}{m}\partial_t$ in $\mathcal{N}_p$.

Proof. We have $[e_1, e_2] = -\tan\theta h_{22}^3 e_2$ because of (32). Thus, if m is a non-vanishing smooth function on M satisfying (36), then we have $[e_1, me_2] = 0$. Therefore, there exists a local coordinate system (s, t) such that $e_1 = \partial_s$ and $e_2 = \dfrac{1}{m}\partial_t$. Thus, the induced metric of M is as given in (35). $\square$

Now, we are ready to obtain the classification theorem.

Theorem 3.6. *Let M be a regular surface in $\mathbb{E}^4$. Let M be a surface endowed with a canonical principal direction relative to $k = (1,0,0,0)$ and assume that the function θ defined in (8) is constant. Then, M is congruent to the surface given by one of the followings*

1. *A surface given by*

$$x(s,t) = s\Big(\cos\theta, \phi_j(t)\sin\theta \Big) + \gamma(t), \quad j = 2,3,4 \qquad (37a)$$

   where γ is the $\mathbb{E}^4$-valued function given by

$$\gamma(t) = \Big(0, \sin\theta \int_{t_0}^t \phi'(\tau)\Psi(\tau)d\tau \Big). \qquad (37b)$$

   Here, $\Psi \in C^\infty(M)$ and ϕ is the unit speed curve lying on $\mathbb{S}^3(1)$ in $\mathbb{E}^4$ such that $\langle \gamma'(t), \phi(t) \rangle = 0$;

2. *A flat surface given by*

$$x(s,t) = s\Big(\cos\theta, \phi_j(t_0)\sin\theta \Big) + \phi(t), \quad j = 2,3,4 \qquad (38)$$

   where $\phi(t_0) = (0, \phi_j(t_0))$ lying on $S^3(1)$ in $\mathbb{E}^4$ is a constant vector perpendicular to the vector $(1,0,0,0)$.

   Conversely, surfaces described above are CPD relative to $k = (1,0,0,0)$.

Proof. Let $\{e_1, e_2; e_3, e_4\}$ be the local orthonormal frame field and coefficients of the second fundamental form described as before in Lemma 3.4, (s,t) a local coordinate system given in Lemma 3.5.

Note that (34a), (34b) and (36) become, respectively

$$(h_{22}^3)_s = -\tan\theta(h_{22}^3)^2, \qquad (39)$$
$$(h_{22}^4)_s + \tan\theta h_{22}^3 h_{22}^4 = 0, \qquad (40)$$
$$m_s - m\tan\theta h_{22}^3 = 0. \qquad (41)$$

Moreover, we have

$$e_1 = x_s. \qquad (42)$$

By combining (41) with (33) we obtain the shape operator S of M as

$$S_3 = \begin{pmatrix} 0 & 0 \\ 0 & \cot\theta\frac{m_s}{m} \end{pmatrix} \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{pmatrix} \qquad (43)$$

where θ is a non-zero constant.

By combining (41) and (39) we get

$$m(s,t) = \Psi_1(t)\Big(s + \Psi_2(t) \Big)$$

for some smooth functions Ψ_1, Ψ_2. Therefore, by re-defining t properly, we may assume either

$$m(s,t) = \sin\theta(s + \Psi(t)), \Psi \in C^\infty(M), \qquad (44a)$$

or

$$m(s,t) = m(t). \tag{44b}$$

Case 1. Let m satisfies (44a). In this case, by considering the equation (32) with (42), we get the Levi-Civita connection of M satisfies

$$\nabla_{\partial_s}\partial_s = 0, \quad \nabla_{\partial_s}\partial_t = \nabla_{\partial_t}\partial_s = \frac{m_s}{m}\partial_t, \quad \nabla_{\partial_t}\partial_t = -mm_s\partial_s + \frac{m_t}{m}\partial_t.$$

By combining these equations with (43) and using Gauss formula (1), we obtain

$$x_{ss} \quad = \quad 0. \tag{45}$$

On the other hand, from the decomposition (8), we have $\langle x_s, k \rangle = \cos\theta$ and $\langle x_t, k \rangle = 0$. By considering these equations, we see that x has the form of

$$x(s,t) = \Big(s\cos\theta, x_j(s,t) + \gamma_j(t) \Big), \quad j = 2, 3, 4. \tag{46}$$

Here $\gamma(t) = (0, \gamma_j(t))$ is a $\mathbb{E}^4$-valued smooth function. On the other hand, since (45) and $\langle x_s, x_s \rangle = 1$, we get $\phi(t)$ is a curve lying on $S^3(1)$ in $\mathbb{E}^4$ with $\phi(t) = (0, \phi_j(t))$. So, if the parametrization reorder, we get

$$x(s,t) = s\Big(\cos\theta, \phi_j(t)\sin\theta \Big) + \gamma(t). \tag{47}$$

Now, by considering $x_{st} = \frac{m_s}{m}x_t$ in (47), we can rewrite the parametrization as

$$x(s,t) = s\Big(\cos\theta, \phi_j(t)\sin\theta \Big) + \sin\theta \int_{t_0}^{t} \Psi(\tau)\phi'(\tau)d\tau, \tag{48}$$

where $\Psi = \Psi(t)$ is a smooth function. Also, since $\langle x_t, x_t \rangle = m^2$, we yield the curve ϕ parameterized by arc-lenght parameter t. Thus, we have the Case (1) of the theorem.

Case 2. m is given as (44b). In this case, the induced metric of M becomes $g = ds^2 + dt^2$, the Levi Civita connection of M satisfies

$$\nabla_{\partial_s}\partial_s = 0, \quad \nabla_{\partial_s}\partial_t = 0, \quad \nabla_{\partial_t}\partial_t = 0. \tag{49}$$

Also, considering (44b) in (34b) and (41), thus (33) becomes

$$S_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{50}$$

where $K(t)$ is a smooth function. Therefore, x and the normal vectors e_3, e_4 satisfy

$$x_{ss} = 0, \quad x_{st} = 0, \quad x_{tt} = e_4.$$
$$(e_3)_s = 0, \quad (e_3)_t = 0,$$
$$(e_4)_s = 0, \quad (e_4)_t = -x_t.$$

A straightforward computation yields that M is congruent to the surface given in Case (2) of the theorem. Hence, the proof for the necessary condition is obtained.

The proof of sufficient condition follows from a direct computation. $\qquad\square$

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