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Research Article

Product of the Generalized L-Subgroups

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We introduce the notion of (λ, μ)-product of L-subsets. We give a necessary and sufficient condition for (λ, μ)-L-subgroup of a product of groups to be (λ, μ)-product of (λ, μ)-L-subgroups.

1. Introduction

Starting from 1980, by the concept of quasi-coincidence of a fuzzy point with a fuzzy set given by Pu and Liu [1], the generalized subalgebraic structures of algebraic structures have been investigated again. Using the concept mentioned above, Bhakat and Das [2, 3] gave the definition of (α, β)-fuzzy subgroup, where α, β are any two of $\{\in, q, \in\vee q, \in\wedge q\}$ with $\alpha \neq \in\wedge q$. The $(\in, \in\vee q)$-fuzzy subgroup is an important and useful generalization of fuzzy subgroups that were laid by Rosenfeld in [4]. After this, many other researchers used the idea of the generalized fuzzy sets that give several characterization results in different branches of algebra (see [5–10]). In recent years, many researchers make generalizations which are referred to as (λ, μ)-fuzzy substructures and $(\in_\lambda, \in_\lambda\vee q_\mu)$-fuzzy substructures on this topic (see [11–15]).

Identifying the subgroups of a Cartesian product of groups plays an essential role in studying group theory. Many important results on characterization of Cartesian product of subgroups, fuzzy subgroups, and TL-fuzzy subgroups exist in literature. Chon obtained a necessary and sufficient condition for a fuzzy subgroup of a Cartesian product of groups to be product of fuzzy subgroups under minimum operation [16]. Later, some necessary and sufficient conditions for a TL-subgroup of a Cartesian product of groups to be a T-product of TL-subgroups were given by Yamak et al. [17]. A subgroup of a Cartesian product of groups is characterized by subgroups in the same study. Consequently, it seems to be interesting to extend this study to generalized L-subgroups. In this paper, we introduce the notion of the (λ, μ)-product of L-subsets and investigate some properties of the (λ, μ)-product of L-subgroups. Also, we give a necessary and sufficient condition for (λ, μ)-L-subgroup of a Cartesian product of groups to be a product of (λ, μ)-L-subgroups.

2. Preliminaries

In this section, we start by giving some known definitions and notations. Throughout this paper, unless otherwise stated, G always stands for any given group with a multiplicative binary operation, an identity e and L denote a complete lattice with top and bottom elements 1, 0, respectively.

An L-subset of X is any function from X into L, which is introduced by Goguen [18] as a generalization of the notion of Zadeh's fuzzy subset [19]. The class of L-subsets of X will be denoted by $F(X, L)$. In particular, if $L = [0, 1]$, then it is appropriate to replace L-subset with fuzzy subset. In this case the set of all fuzzy subsets of X is denoted by $F(X)$. Let A and B be L-subsets of X. We say that A is contained in B if $A(x) \leq B(x)$ for every $x \in X$ and is denoted by $A \leq B$. Then $\leq$ is a partial ordering on the set $F(X, L)$.

Definition 1 (see [20]). An L-subset of G is called an L-subgroup of G if, for all $x, y \in G$, the following conditions hold:

(G1) $A(x) \wedge A(y) \leq A(xy)$.

(G2) $A(x) \leq A(x^{-1})$.

In particular, when $L = [0, 1]$, an L-subgroup of G is referred to as a fuzzy subgroup of G.

Definition 2 (see [12]). Let $\lambda, \mu \in L$ and $\lambda < \mu$. Let A be an L-subset of G. A is called (λ, μ)-L-subgroup of G if, for all $x, y \in G$, the following conditions hold:

(i) $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$.

(ii) $A(x^{-1}) \vee \lambda \geq A(x) \wedge \mu$.

Denote by $FS(\lambda, \mu, G, L)$ the set of all (λ, μ)-L-subgroups of G. When $L = [0, 1]$, its counterpart is written as $FS(\lambda, \mu, G)$.

Unless otherwise stated, L always represents any given distributive lattice.

3. Product of (λ,μ)-L-Subgroups

Definition 3 (see [16]). Let A_i be an L-subset of G_i for each $i = 1, 2, \ldots, n$. Then product of A_i $(i = 1, 2, \ldots, n)$ denoted by $A_1 \times A_2 \times \cdots \times A_n$ is defined to be the L-subset of $G_1 \times G_2 \times \cdots \times G_n$ that satisfies

$$(A_1 \times A_2 \times \cdots \times A_n)(x_1, x_2, \ldots, x_n) = A_1(x_1) \wedge A_2(x_2) \wedge \cdots \wedge A_n(x_n). \tag{1}$$

Example 4. We define the fuzzy subsets A and B of $\mathbb{Z}$ and $\mathbb{Z}_2$, respectively, as follows:

$$A(x) = \begin{cases} 0.5, & x \in 2\mathbb{Z}, \\ 0.3, & \text{otherwise}, \end{cases} \qquad B(x) = \begin{cases} 0.7, & x = \overline{0}, \\ 0.2, & x = \overline{1}. \end{cases} \tag{2}$$

We obtain that

$$A \times B(x) = \begin{cases} 0.5, & x \in 2\mathbb{Z} \times \{\overline{0}\}, \\ 0.3, & x \in \mathbb{Z} - \{2\mathbb{Z}\} \times \{\overline{0}\}, \\ 0.2, & \text{otherwise}. \end{cases} \tag{3}$$

Theorem 5. *Let $G_1, G_2, \ldots, G_n$ be groups and $\bigvee_{i=1}^{n} \lambda_i < \bigwedge_{i=1}^{n} \mu_i$ such that A_i is (λ_i, μ_i)-L-subgroup of G_i for each $i = 1, 2, \ldots, n$. Then $A_1 \times A_2 \times \cdots \times A_n$ is $(\bigvee_{i=1}^{n} \lambda_i, \bigwedge_{i=1}^{n} \mu_i)$-$L$-subgroup of $G_1 \times G_2 \times \cdots \times G_n$.*

Proof. Let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in G_1 \times G_2 \times \cdots \times G_n$. Then

$$\begin{aligned} & A_1 \times A_2 \times \cdots \times A_n((x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n)) \\ & \quad \vee \bigvee_{i=1}^{n} \lambda_i = A_1 \times A_2 \times \cdots \\ & \quad \times A_n(x_1 y_1, x_2 y_2, \ldots, x_n y_n) \vee \lambda_1 \vee \bigvee_{i=1}^{n} \lambda_i \\ & \quad = (A_1(x_1 y_1) \wedge A_2(x_2 y_2) \wedge \cdots \wedge A_n(x_n y_n)) \\ & \quad \vee \bigvee_{i=1}^{n} \lambda_i \geq (A_1(x_1 y_1) \vee \lambda_1) \wedge (A_2(x_2 y_2) \vee \lambda_2) \\ & \quad \wedge \cdots \wedge (A_n(x_n y_n) \vee \lambda_n) \geq A_1(x_1) \wedge A_1(y_1) \wedge \mu_1 \\ & \quad \wedge A_2(x_2) \wedge A_2(y_2) \wedge \mu_2 \wedge \cdots \wedge A_n(x_n) \wedge A_n(y_n) \\ & \quad \wedge \mu_n = A_1 \times A_2 \times \cdots \times A_n(x_1, x_2, \ldots, x_n) \wedge A_1 \\ & \quad \times A_2 \times \cdots \times A_n(y_1, y_2, \ldots, y_n) \wedge \bigwedge_{i=1}^{n} \mu_i. \end{aligned} \tag{4}$$

Similarly, it can be shown that

$$\begin{aligned} & A_1 \times A_2 \times \cdots \times A_n((x_1, x_2, \ldots, x_n)^{-1}) \vee \bigvee_{i=1}^{n} \lambda_i \\ & \quad \geq A_1 \times A_2 \times \cdots \times A_n(x_1, x_2, \ldots, x_n) \wedge \bigwedge_{i=1}^{n} \mu_i. \end{aligned} \tag{5}$$

□

Corollary 6. *If $A_1, A_2, \ldots, A_n$ are (λ, μ)-L-subgroups of $G_1, G_2, \ldots, G_n$, respectively, then $A_1 \times A_2 \times \cdots \times A_n$ is (λ, μ)-L-subgroup of $G_1 \times G_2 \times \cdots \times G_n$.*

Theorem 7 (Theorem 2.9, [16]). *Let $G_1, G_2, \ldots, G_n$ be groups, let $e_1, e_2, \ldots, e_n$ be identities, respectively, and let A be a fuzzy subgroup in $G_1 \times G_2 \times \cdots \times G_n$. Then $A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \geq A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, n$ if and only if $A = A_1 \times A_2 \times \cdots \times A_n$, where $A_1, A_2, \ldots, A_n$ are fuzzy subgroups of $G_1, G_2, \ldots, G_n$, respectively.*

The following example shows that Theorem 7 may not be true for any (λ, μ)-L-subgroup.

Example 8. Consider

$$A(x) = \begin{cases} 0.8, & x = (\overline{0}, \overline{0}), \\ 0.7, & x = (\overline{1}, \overline{0}), \\ 0.6, & x = (\overline{0}, \overline{1}), \\ 0.5, & x = (\overline{1}, \overline{1}). \end{cases} \tag{6}$$

It is easy to see that A is $(0, 0.5)$-fuzzy subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since $A(\overline{1}, \overline{0}) \geq A(\overline{1}, \overline{1})$ and $A(\overline{0}, \overline{1}) \geq A(\overline{1}, \overline{1})$, A satisfies the condition of Theorem 7, but there do not exist A_1, A_2 fuzzy subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ such that $A = A_1 \times A_2$.

In fact, suppose that there exist $A_1, A_2 \in FS(0, 0.5, \mathbb{Z}_2 \times \mathbb{Z}_2)$ such that $A = A_1 \times A_2$. Since $A(\overline{1}, \overline{0}) = 0.7$ and $A(\overline{0}, \overline{1}) = 0.6$, we have $A_1(\overline{1}) \geq 0.7$ and $A_2(\overline{1}) \geq 0.6$. Hence

$$0.5 = A_1 \times A_2 \left(\overline{1}, \overline{1}\right) = A_1\left(\overline{1}\right) \wedge A_2\left(\overline{1}\right) \geq 0.7 \wedge 0.6 = 0.6, \tag{7}$$

a contradiction.

Definition 9. Let A_i be an L-subset of G_i for each $i = 1, 2, \ldots, n$. Then (λ, μ)-product of A_i $(i = 1, 2, \ldots, n)$ denoted by $A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n$ is defined to be an L-subset of $G_1 \times G_2 \times \cdots \times G_n$ that satisfies

$$\begin{aligned}\left(A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n\right)\left(x_1, x_2, \ldots, x_n\right) &= \left(A_1\left(x_1\right) \wedge A_2\left(x_2\right) \wedge \cdots \wedge A_n\left(x_n\right) \wedge \mu\right) \vee \lambda \\ &= \left(\bigwedge_{i=1}^{n} A_i\left(x_i\right) \wedge \mu\right) \vee \lambda.\end{aligned} \tag{8}$$

Example 10. We define the fuzzy subsets A and B of $\mathbb{Z}$ and $\mathbb{Z}_2$, respectively, as in Example 4. Then $(0.4, 0.6)$-product of A and B is as follows:

$$A \times_{0.6}^{0.4} B(x) = \begin{cases} 0.5, & x \in 2\mathbb{Z} \times \left\{\overline{0}\right\}, \\ 0.4, & \text{otherwise}. \end{cases} \tag{9}$$

Lemma 11. *Let $G_1, G_2, \ldots, G_n$ be groups. Then we have the following:*

(1) *If A is (λ, μ)-L-subgroup of $G_1 \times G_2 \times, \ldots, G_n$ and $A_i(x) = A(e_1, e_2, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n)$ for $i = 1, 2, \ldots, n$, then A_i is (λ, μ)-L-subgroup of G_i for all $i = 1, 2, \ldots, n$.*

(2) *If A_i is (λ, μ)-L-subgroup of G_i for all $i = 1, 2, \ldots, n$, then $A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n$ is (λ, μ)-L-subgroup of $G_1 \times G_2 \times, \ldots, G_n$.*

Proof. (1) Let $x, y \in G_i$. Since $A \in FS(\lambda, \mu, G_1 \times G_2 \times, \ldots, G_n, L)$, we have

$$\begin{aligned}A_i(x) \wedge A_i(y) \wedge \mu &= A\left(e_1, e_2, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n\right) \\ &\quad \wedge A\left(e_1, e_2, \ldots, e_{i-1}, y, e_{i+1}, \ldots, e_n\right) \wedge \mu \\ &\leq A\left(e_1, e_2, \ldots, e_{i-1}, x \cdot y, e_{i+1}, \ldots, e_n\right) \vee \lambda \\ &= A_i(x \cdot y) \vee \lambda.\end{aligned} \tag{10}$$

Similarly, we can show that $A_i(x) \wedge \mu \leq A_i(x^{-1}) \vee \lambda$. Hence, $A_i \in FS(\lambda, \mu, G_i, L)$ by Definition 2 for $i = 1, 2, \ldots, n$.

(2) Let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in G_1 \times G_2 \times, \ldots, G_n$. Then,

$$\begin{aligned}&A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n\left(\left(x_1, x_2, \ldots, x_n\right)\left(y_1, y_2, \ldots, y_n\right)\right) \\ &\quad \vee \lambda = A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n\left(x_1 y_1, x_2 y_2, \ldots, x_n y_n\right) \\ &\quad \vee \lambda = \left(\bigwedge_{i=1}^{n} A_i\left(x_i y_i\right) \wedge \mu\right) \vee \lambda = \left(\left(A_1\left(x_1 y_1\right) \vee \lambda\right)\right. \\ &\quad \wedge \left(A_2\left(x_2 y_2\right) \vee \lambda\right) \wedge \cdots \wedge \left(A_n\left(x_n y_n\right) \vee \lambda\right) \wedge \mu) \\ &\quad \vee \lambda \geq \left(A_1\left(x_1\right) \wedge A_1\left(y_1\right) \wedge \mu \wedge A_2\left(x_2\right) \wedge A_2\left(y_2\right)\right. \\ &\quad \wedge \mu \wedge \cdots \wedge A_n\left(x_n\right) \wedge A_n\left(y_n\right) \wedge \mu) \vee \lambda \\ &\quad = \left(\left(\bigwedge_{i=1}^{n} A_i\left(x_i\right) \wedge \mu\right) \vee \lambda\right) \wedge \left(\left(\bigwedge_{i=1}^{n} A_i\left(y_i\right) \wedge \mu\right)\right. \\ &\quad \left.\vee \lambda\right) \wedge \mu = A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n\left(x_1, x_2, \ldots, x_n\right) \\ &\quad \wedge A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n\left(y_1, y_2, \ldots, y_n\right) \wedge \mu.\end{aligned} \tag{11}$$

Similarly, we can show that

$$\begin{aligned}&A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n\left(\left(x_1, x_2, \ldots, x_n\right)^{-1}\right) \vee \lambda \\ &\quad \geq A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n\left(x_1, x_2, \ldots, x_n\right) \wedge \mu.\end{aligned} \tag{12}$$

Hence, $A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n$ is (λ, μ)-L-subgroup of $G_1 \times G_2 \times, \ldots, G_n$. □

Theorem 12. *Let $G_1, G_2, \ldots, G_n$ be groups and $A \in FS(\lambda, \mu, G_1 \times G_2 \times \cdots \times G_n, L)$. Then $(A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \wedge \mu) \vee \lambda = A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, n$ if and only if $A = A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n$, where $A_i(x) = A(e_1, e_2, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n)$.*

Proof. Suppose that $(A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \wedge \mu) \vee \lambda = A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, n$. Now, for any $(x_1, x_2, \ldots, x_n) \in G_1 \times G_2 \times \cdots \times G_n$, we have

$$\begin{aligned}A\left(x_1, x_2, \ldots, x_n\right) &= \left(A\left(x_1, e_2, \ldots, e_n\right)\right. \\ &\quad \left.\wedge A\left(e_1, x_2, \ldots, x_n\right) \wedge \mu\right) \vee \lambda \\ &= \left(A_1\left(x_1\right)\right. \\ &\quad \wedge \left(\left(A\left(e_1, x_2, \ldots, e_n\right) \wedge A\left(e_1, e_2, x_3, \ldots, x_n\right) \wedge \mu\right)\right. \\ &\quad \left.\left.\vee \lambda\right) \wedge \mu\right) \vee \lambda\end{aligned}$$

$$\begin{aligned}
&= (A_1(x_1) \wedge A_2(x_2) \wedge A(e_1, e_2, x_3, \ldots, x_n) \wedge \mu) \\
&\quad \vee \lambda \\
&\qquad \vdots \\
&= (A_1(x_1) \wedge A_2(x_2) \wedge \cdots \wedge A(e_1, e_2, \ldots, x_n) \wedge \mu) \\
&\quad \vee \lambda \\
&= A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n (x_1, x_2, \ldots, x_n).
\end{aligned} \tag{13}$$

Hence we obtain that $A = A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n$. Conversely, assume that $A = A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \cdots \times_\mu^\lambda A_n$. Then, we obtain

$$\begin{aligned}
A(x_1, x_2, \ldots, x_n) &= A_1 \times_\mu^\lambda A_2 \times_\mu^\lambda \\
&\quad \cdots \times_\mu^\lambda A_n (x_1, x_2, \ldots, x_n) \\
&= (A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, e_n) \wedge \cdots \\
&\quad \wedge A(e_1, e_2, \ldots, x_n) \wedge \mu) \vee \lambda \\
&\le (A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, e_n) \wedge \cdots \\
&\quad \wedge (A(e_1, e_2, \ldots, x_{n-1}, x_n) \vee \lambda) \wedge \mu) \vee \lambda \\
&= (A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, e_n) \wedge \cdots \\
&\quad \wedge A(e_1, e_2, \ldots, x_{n-1}, x_n) \wedge \mu) \vee \lambda \\
&\qquad \vdots \\
&\le (A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu) \vee \lambda.
\end{aligned} \tag{14}$$

On the other hand, $(A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu) \vee \lambda \le A(x_1, x_2, \ldots, x_n) \vee \lambda$.

Since $A(x_1, x_2, \ldots, x_n) \ge \lambda$, $(A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu) \vee \lambda \le A(x_1, x_2, \ldots, x_n)$.

Hence, $A(x_1, x_2, \ldots, x_n) = (A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu) \vee \lambda$.

Similarly, we get $(A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \wedge \mu) \vee \lambda = A(x_1, x_2, \ldots, x_n)$ for $i = 2, 3, \ldots, n$. □

Lemma 13. *Let $G_1, G_2, \ldots, G_n$ be groups, let $e_1, e_2, \ldots, e_n$ be identities of $G_1, G_2, \ldots, G_n$, respectively, and $A \in FS(\lambda, \mu, G_1 \times G_2 \times \cdots \times G_n)$. If $(A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge \mu) \vee \lambda \ge A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, n$, then $(A(e_1, e_2, \ldots, e_{k-1}, x_k, e_{k+1}, \ldots, e_n) \wedge \mu) \vee \lambda \ge A(x_1, x_2, \ldots, x_n)$.*

Proof. By Definition 2, we observe that

$$\begin{aligned}
&(A(e_1, e_2, \ldots, e_{k-1}, x_k, e_{k+1}, \ldots, e_n) \wedge \mu) \vee \lambda \\
&\quad = \left(A\left((x_1, x_2, \ldots, x_n) \cdot \left(x_1^{-1}, x_2^{-1}, \ldots, x_{k-1}^{-1}, e_k, x_{k+1}^{-1}, \ldots, x_n^{-1}\right)\right) \wedge \mu\right) \vee \lambda \\
&\quad = \left(A\left((x_1, x_2, \ldots, x_n) \cdot \left(x_1^{-1}, x_2^{-1}, \ldots, x_{k-1}^{-1}, e_k, x_{k+1}^{-1}, \ldots, x_n^{-1}\right)\right) \vee \lambda\right) \wedge \mu \\
&\quad \ge \left(A(x_1, x_2, \ldots, x_n) \wedge A\left(x_1^{-1}, x_2^{-1}, \ldots, x_{k-1}^{-1}, e_k, x_{k+1}^{-1}, \ldots, x_n^{-1}\right) \wedge \mu\right) \vee \lambda \\
&\quad = (A(x_1, x_2, \ldots, x_n) \vee \lambda) \wedge \left(A\left(x_1^{-1}, x_2^{-1}, \ldots, x_{k-1}^{-1}, e_k, x_{k+1}^{-1}, \ldots, x_n^{-1}\right) \vee \lambda\right) \wedge \mu \\
&\quad \ge ((A(x_1, x_2, \ldots, x_n) \vee \lambda) \wedge A(x_1, x_2, \ldots, x_{k-1}, e_k, x_{k+1}, \ldots, x_n) \wedge \mu) \vee \lambda \\
&\quad = (A(x_1, x_2, \ldots, x_n) \vee \lambda) \wedge (A(x_1, x_2, \ldots, x_{k-1}, e_k, x_{k+1}, \ldots, x_n) \vee \lambda) \wedge \mu \\
&\quad \ge ((A(x_1, x_2, \ldots, x_n) \vee \lambda) \wedge (A(e_1, \ldots, e_{k-2}, x_{k-1}, e_k, \ldots, e_n) \wedge A(x_1, \ldots, e_{k-1}, e_k, x_{k+1}, \ldots, x_n) \wedge \mu)) \vee \lambda \\
&\quad = (A(x_1, x_2, \ldots, x_n) \vee \lambda) \wedge (A(x_1, \ldots, e_{k-1}, e_k, x_{k+1}, \ldots, x_n) \wedge \mu) \vee \lambda \\
&\qquad \vdots \\
&\quad \ge A(x_1, x_2, \ldots, x_n) \vee \lambda \\
&\quad \ge A(x_1, x_2, \ldots, x_n).
\end{aligned} \tag{15}$$

□

Lemma 14. *Let $G_1, G_2, \ldots, G_n$ be groups and $A \in FS(0, \mu, G_1 \times G_2 \times \cdots \times G_n, L)$. Then $A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \wedge \mu = A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, n$ if and only if $A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge \mu \ge A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, n$.*

Proof. Now assume that $A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \wedge \mu = A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, n$. Next, for any $(x_1, x_2, \ldots, x_n) \in G_1 \times G_2 \times \cdots \times G_n$, we have

$$\begin{aligned}
A(x_1, x_2, \ldots, x_n) &= A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \\
&\quad \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \wedge \mu \\
&\le A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge \mu.
\end{aligned} \tag{16}$$

Conversely, assume that $A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge \mu \ge A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, n$. By

Definition 2 and Lemma 11, we obtain that

$$
\begin{aligned}
A(x_1, x_2, \ldots, x_n) &= A(x_1, x_2, \ldots, x_n) \\
&\quad \wedge A(x_1, x_2, \ldots, x_n) \wedge \cdots \wedge A(x_1, x_2, \ldots, x_n) \\
&\leq A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, e_n) \wedge \cdots \\
&\quad \wedge A(e_1, e_2, \ldots, x_n) \wedge \mu \\
&\leq A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, e_n) \wedge \cdots \\
&\quad \wedge A(e_1, e_2, \ldots, x_{n-1}, x_n) \wedge \mu \\
&\quad \vdots \\
&\leq A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu \\
&\leq A(x_1, x_2, \ldots, x_n).
\end{aligned}
\tag{17}
$$

Hence, $A(x_1, x_2, \ldots, x_n) = A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu$. Similarly, we get $A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) = A(x_1, x_2, \ldots, x_n)$ for $i = 2, 3, \ldots, n$. □

As a consequence of Theorem 12 and Lemma 14, we have the following corollary.

Corollary 15. *Let $G_1, G_2, \ldots, G_n$ be groups and $A \in FS(0, \mu, G_1 \times G_2 \times \cdots \times G_n, L)$. Then $A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \wedge \mu \geq A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, k-1, k+1, \ldots, n$ if and only if $A = A_1 \times_\mu A_2 \times_\mu \cdots \times_\mu A_n$, where $A_i(x) = A(e_1, e_2, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n)$.*

The following example shows that Corollary 15 may not be true when $\lambda \neq 0$.

Example 16. Consider

$$
A(x) = \begin{cases} 0.4, & x = (\overline{0}, \overline{0}), \\ 0.3, & x = (\overline{1}, \overline{0}), \\ 0.2, & x = (\overline{0}, \overline{1}), \\ 0.1, & x = (\overline{1}, \overline{1}). \end{cases} \tag{18}
$$

A is $(0.2, 0.5)$-fuzzy subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and satisfies the necessary condition of Corollary 15. But there is not any A_1 and A_2, $(0.2, 0.5)$-fuzzy subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which hold $A = A_1 \times_{0.5}^{0.2} A_2$.

4. Conclusion

In this study, we give a necessary and sufficient condition for $(0, \mu)$-L-subgroup of a Cartesian product of groups to be a product of $(0, \mu)$-L-subgroups. The results obtained are not valid for $\lambda \neq 0$, and a counterexample is provided.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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