

THEOREMS ON GENOCCHI POLYNOMIALS OF HIGHER ORDER ARISING FROM GENOCCHI BASIS

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Abstract. Recently, Kim *et al.* [8] constructed a new method to obtain interesting identities related to Euler polynomials of higher order arising from Euler basis. In the present paper, we study to Genocchi polynomials of higher order arising from Genocchi basis by using the method of Kim *et al.* We also derive many interesting properties related to Genocchi polynomials of higher order.

1. INTRODUCTION

In the complex plane, the Genocchi polynomials are defined by the following condition:

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = e^{tG(x)} = \frac{2t}{e^t + 1} e^{xt}, \quad (|t| < \pi)$$

with the usual convention of replacing $(G(x))^n := G_n(x)$, is used (see [3, 6, 10, 12, 13, 14, 17]).

As is well known, the Genocchi polynomials of order k are defined via the generating function to be

$$(1.1) \quad \left(\frac{2t}{e^t + 1} \right)^k e^{xt} = e^{tG^{(k)}(x)} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!} \quad (k \in \mathbf{Z}_+ = \mathbb{N} \cup \{0\}),$$

with the usual convention about replacing $(G^{(k)}(x))^n$ by $G_n^{(k)}(x)$ (see [14, 17]).

In the special case, $x = 0$, $G_n^{(k)}(0) := G_n^{(k)}$ are called the Genocchi numbers of order k .

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By (1.1), we easily get

$$(1.2) \quad \begin{aligned} G_n^{(k)}(x) &= \sum_{l=0}^n \binom{n}{l} G_l^{(k)} x^{n-l} = \sum_{l=0}^n \binom{n}{l} G_{n-l}^{(k)} x^l \\ &= \sum_{n=n_1+\dots+n_r} \binom{n}{n_1, \dots, n_r} G_{n_1} G_{n_2} \cdots G_{n_{r-1}} x^{n_r}. \end{aligned}$$

From (1.1), we have

$$G_n^{(0)}(x) = x^n.$$

By (1.1), it is not difficult to show that

$$\frac{dG_n^{(k)}(x)}{dx} = nG_{n-1}^{(k)}(x) \text{ and } G_n^{(k)}(x+1) + G_n^{(k)}(x) = 2nG_{n-1}^{(k-1)}(x), \quad (\text{see [14, 17]}).$$

We now define two linear operators Λ and D on the space of real-valued differentiable functions as follows: for $n \in \mathbb{N}$

$$(1.3) \quad \Lambda f_n(x) = \frac{f_n(x+1) + f_n(x)}{n} \text{ and } Df_n(x) = \frac{df_n(x)}{dx}.$$

We note that $\Lambda D = D\Lambda$. By (1.3), we have

$$\Lambda^2 f_n(x) = \Lambda(\Lambda f_n(x)) = \frac{1}{n^2} \sum_{l=0}^2 \binom{2}{l} f_n(x+l).$$

Continuing this process, we get

$$\Lambda^k f_n(x) = \frac{1}{n^k} \sum_{l=0}^k \binom{k}{l} f_n(x+l).$$

Therefore, we acquire the following Lemma.

Lemma 1. *Let f_n be real valued function and $k \in \mathbb{N}$, then we have*

$$\Lambda^k f(x) = \frac{1}{n^k} \sum_{l=0}^k \binom{k}{l} f(x+l).$$

Obviously,

$$(1.4) \quad \Lambda^k f(0) = \frac{1}{n^k} \sum_{l=0}^k \binom{k}{l} f(l).$$

Let $T_n = \{q_n(x) \in \mathbb{Q}[x] \mid \deg q_n(x) \leq n\}$ be the $(n + 1)$ -dimensional vector space over \mathbb{Q} . Probably, $\{1, x, \dots, x^n\}$ is the most natural basis for T_n . But $\{G_k^{(k)}, G_{k+1}^{(k)}, \dots, G_{n+k}^{(k)}\}$ is also a good basis for the space T_n for our objective of arithmetical and combinatorial applications of the Genocchi polynomials of higher order.

If $q_n(x) \in T_n$, then $q_n(x)$ can be expressed by

$$q_n(x) = b_k G_k^{(k)}(x) + \dots + b_{n+k} G_{n+k}^{(k)}(x) = \sum_{l=k}^{n+k} b_l G_l^{(k)}(x).$$

In the present paper, we develop methods for computing b_l from the information of $q_n(x)$ and apply those results to arithmetically and combinatorially interesting identities involving $G_k^{(k)}, \dots, G_{n+k}^{(k)}$.

2. SOME IDENTITIES ON THE GENOCCHI POLYNOMIALS OF HIGHER ORDER

By (1.3), we have

$$(2.1) \quad \Lambda G_n^{(k)}(x) = \frac{G_n^{(k)}(x+1) + G_n^{(k)}(x)}{n} = 2G_{n-1}^{(k-1)}(x),$$

and

$$(2.2) \quad DG_n^{(k)}(x) = nG_{n-1}^{(k)}(x).$$

Let us suppose that $q_n(x) \in T_n$. Then $q_n(x)$ can be generated by $G_k^{(k)}(x), G_{k+1}^{(k)}(x), \dots, G_{n+k}^{(k)}(x)$ as follows:

$$(2.3) \quad q_n(x) = \sum_{l=k}^{n+k} b_l G_l^{(k)}(x).$$

Thus, by (2.3), we get

$$\Lambda q_n(x) = \sum_{l=k}^{n+k} b_l \Lambda G_l^{(k)}(x) = 2 \sum_{l=k}^{n+k} b_l G_{l-1}^{(k-1)}(x),$$

and

$$\Lambda^2 q_n(x) = 2 \sum_{l=k}^{n+k} b_l \Lambda G_{l-1}^{(k-1)}(x) = 2^2 \sum_{l=k}^{n+k} b_l G_{l-2}^{(k-2)}(x).$$

Continuing this process, we have

$$(2.4) \quad \Lambda^k q_n(x) = 2^k \sum_{l=k}^{n+k} b_l G_{l-k}^{(0)}(x) = 2^k \sum_{l=k}^{n+k} b_l x^{l-k}.$$

Let us apply the operator D^j on (2.4). Then

$$(2.5) \quad \begin{aligned} D^j \Lambda^k q_n(x) &= 2^k \sum_{l=k+j}^{n+k} b_l \prod_{a=0}^{j-1} (l-k-a) x^{l-k-j} \\ &= 2^k \sum_{l=k+j}^{n+k} b_l \frac{(l-k)!}{(l-k-j)!} x^{l-k-j} \\ &= 2^k \sum_{l=k+j}^{n+k} b_l j! \binom{l-k}{j} x^{l-k-j}. \end{aligned}$$

Let us take $x = 0$ on (2.5). Then we get

$$(2.6) \quad D^j \Lambda^k q_n(0) = 2^k b_{k+j} j!.$$

From (1.4) and (2.6), we have

$$(2.7) \quad \begin{aligned} b_{k+j} &= \frac{1}{2^k j!} D^j \Lambda^k q_n(0) = \frac{1}{2^k j!} \Lambda^k D^j q_n(0) \\ &= \frac{1}{2^k j! n^k} \sum_{m=0}^k \binom{k}{m} D^j q_n(m). \end{aligned}$$

Therefore, by (2.3) and (2.7), we procure the following theorem.

Theorem 1. For $k, n \in \mathbb{Z}_+$ and $q_n(x) \in T_n$, then we have

$$q_n(x) = \frac{1}{(2n)^k} \sum_{l=k}^{n+k} \left(\frac{1}{(l-k)!} \sum_{m=0}^k \binom{k}{m} D^{l-k} q_n(m) \right) G_l^{(k)}(x).$$

Let us take $q_n(x) = x^n \in T_n$. Then we readily derive that $D^k x^n = \frac{n!}{(n-k)!} x^{n-k}$.

Thus, by Theorem 1, we get

$$(2.8) \quad \begin{aligned} x^n &= \frac{1}{(2n)^k} \sum_{l=k}^{n+k} \left(\frac{1}{(l-k)!} \sum_{m=0}^k \binom{k}{m} \frac{n!}{(n-l+k)!} m^{n-l+k} \right) G_l^{(k)}(x) \\ &= \frac{1}{(2n)^k} \sum_{l=k}^{n+k} \sum_{m=0}^k \binom{k}{m} \binom{n}{l-k} m^{n-l+k} G_l^{(k)}(x). \end{aligned}$$

Therefore, by (2.8), we get the following corollary.

Corollary 1. For $k, j, n \in \mathbb{Z}_+$, then we have

$$x^n = \frac{1}{(2n)^k} \sum_{l=k}^{n+k} \sum_{m=0}^k \binom{k}{m} \binom{n}{l-k} m^{n-l+k} G_l^{(k)}(x).$$

In [6, 18] and [20], Bernoulli polynomials of higher order are defined by the rule:

$$\sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} = \left(\frac{t}{e^t - 1} \right)^k e^{xt}, \quad (|t| < 2\pi).$$

Let $q_n(x) = B_n^{(k)}(x) \in T_n$. Also, it is well known in [6] that

$$(2.9) \quad D^{l-k} B_n^{(k)}(x) = \frac{n!}{(n-l+k)!} B_{n-l+k}^{(k)}(x).$$

By Theorem 1 and (2.9), we get

$$(2.10) \quad B_n^{(k)}(x) = \frac{1}{(2n)^k} \sum_{l=k}^{n+k} \left(\sum_{m=0}^k \binom{k}{m} \binom{n}{l-k} B_{n-l+k}^{(k)}(m) \right) G_l^{(k)}(x).$$

Therefore, by (2.10), we discover the following theorem.

Theorem 2. For $k, n \in \mathbb{Z}_+$, then we have

$$B_n^{(k)}(x) = \frac{1}{(2n)^k} \sum_{l=k}^{n+k} \left(\sum_{m=0}^k \binom{k}{m} \binom{n}{l-k} B_{n-l+k}^{(k)}(m) \right) G_l^{(k)}(x).$$

In [5, 8, 9] and [14], it is well known that

$$\left(\frac{2}{e^t + 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}, \quad (|t| < \pi)$$

where $E_n^{(k)}(x)$ are called Euler polynomials of higher order.

Let us consider $q_n(x) = E_n^{(k)}(x) \in T_n$. Then we see that

$$(2.11) \quad D^{l-k} E_n^{(k)}(x) = \frac{n!}{(n-l+k)!} E_{n-l+k}^{(k)}(x).$$

On account of Theorem 1 and (2.11), we get the following theorem.

Theorem 3. For $k, n \in \mathbb{Z}_+$, then we have

$$E_n^{(k)}(x) = \frac{1}{(2n)^k} \sum_{l=k}^{n+k} \left(\sum_{m=0}^k \binom{k}{m} \binom{n}{l-k} E_{n-l+k}^{(k)}(m) \right) G_l^{(k)}(x).$$

Hansen [19] derived the following convolution formula:

$$\sum_{k=0}^m \binom{m}{k} B_k(x) B_{m-k}(y) = (1-m) B_m(x+y) + (x+y-1) m B_{m-1}(x+y).$$

We want to note that the special case $x = y = 0$ of the last identity:

$$\sum_{k=2}^{m-2} \binom{m}{k} B_k B_{m-k} = -(m+1) B_m$$

is originally constructed by Euler and Ramanujan (cf. [18]).

Let us now consider the following expression for a fixed y

$$(2.12) \quad q_n(x) = \sum_{k=0}^n \binom{n}{k} B_k(x) B_{n-k}(y) \in T_n.$$

Therefore, we derive the following equality:

$$(2.13) \quad \begin{aligned} D^j q_n(x) &= (1-n) \frac{n!}{(n-j)!} B_{n-j}(x+y) \\ &+ (x+y-1) \frac{n!}{(n-j-1)!} B_{n-j-1}(x+y) + j \frac{n!}{(n-j)!} B_{n-j}(x+y). \end{aligned}$$

From Theorem 1, (2.12) and (2.13), we obtain the following theorem.

Theorem 4. For $k, n \in \mathbb{Z}_+$ and a fixed y , then we have

$$\begin{aligned} &\sum_{k=0}^n \binom{n}{k} B_k(x) B_{n-k}(y) \\ &= \frac{1}{(2n)^k} \sum_{l=k}^{n+k} \sum_{m=0}^k \binom{k}{m} \left\{ (1-n) \binom{n}{l-k} B_{n-l+k}(m+y) \right. \\ &\quad \left. + n(m+y-1) \binom{n-1}{l-k} B_{n-l+k-1}(m+y) \right. \\ &\quad \left. + (l-k) \binom{n}{l-k} B_{n-l+k}(m+y) \right\} G_l^{(k)}(x). \end{aligned}$$

For $\lambda (\neq 1) \in \mathbb{C}$, the Frobenius-Euler polynomials of order k are also known by [1, 2, 15] and [21]

$$(2.14) \quad \underbrace{\frac{1-\lambda}{e^t-\lambda} \cdots \frac{1-\lambda}{e^t-\lambda}}_{k\text{-times}} e^{xt} = \left(\frac{1-\lambda}{e^t-\lambda} \right)^k e^{xt} = \sum_{n=0}^{\infty} H_n^{(k)}(x|\lambda) \frac{t^n}{n!}.$$

Let $q_n(x) = H_n^{(k)}(x | \lambda) \in T_n$. We readily derive that

$$(2.15) \quad D^{l-k} H_n^{(k)}(x | \lambda) = \frac{n!}{(n-l+k)!} H_{n-l+k}^{(k)}(x | \lambda).$$

Therefore, by Theorem 1, (2.14) and (2.15), then we get the following theorem.

Theorem 5. *The following identity holds true:*

$$H_n^{(k)}(x | \lambda) = \frac{1}{(2n)^k} \sum_{l=k}^{n+k} \sum_{m=0}^k \binom{k}{m} \binom{n}{l-k} H_{n-l+k}^{(k)}(x | \lambda) G_l^{(k)}(x).$$

Dilcher [18] derived the following both interesting and fascinating convolution formula:

$$\sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y) = 2(1-x-y) E_n(x+y) + 2E_{n+1}(x+y).$$

Thus, we easily see that $\sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y) \in T_n$. So, we consider for a fixed y

$$(2.16) \quad q_n(x) = \sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y).$$

It is not difficult to see the following:

$$D^j q_n(x) = 2 \left\{ \frac{n!}{(n-j)!} (1-x-y) E_{n-j}(x+y) - j \frac{n!}{(n-j+1)!} \times E_{n-j+1}(x+y) + \frac{(n+1)!}{(n+1-j)!} E_{n+1-j}(x+y) \right\}.$$

As a result of the last identity and Theorem 1, the following theorem can be stated.

Theorem 6. *For a fixed y , then we have*

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} E_k(x) E_{n-k}(y) \\ &= \frac{2}{(2n)^k} \sum_{l=k}^{n+k} \sum_{m=0}^k \binom{k}{m} \left\{ \binom{n}{l-k} E_{n-l+k}(m+y) \right. \\ & \quad \left. - \frac{l+k}{n+1} \binom{n+1}{l-k} E_{n-l+k+1}(m+y) + \binom{n+1}{l-k} E_{n+1-l+k}(m+y) \right\} G_l^{(k)}(x). \end{aligned}$$

In the complex plane, the Hermite polynomials, named after Charles Hermite, are given by the exponential generating function:

$$(2.17) \quad \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2tx-t^2}.$$

One can also derive the generating function of Hermite polynomials by using Cauchy's integral formula as

$$(2.18) \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = (-1)^n e^{x^2} \frac{n!}{2\pi i} \int_C \frac{e^{z^2}}{(z-x)^{n+1}} dz,$$

where C is a loop which starts at $-\infty$, encircles the origin once in the positive direction, and the returns $-\infty$ (see [7]).

Let $q_n(x) = H_n(x) \in T_n$. Thus, from (2.17), we have

$$(2.19) \quad D^k q_n(x) = 2^k \frac{n!}{(n-k)!} H_{n-k}(x).$$

Therefore, by Theorem 1, (2.18) and (2.19), we arrive at the following theorem.

Theorem 7. *The following identity holds true:*

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} \frac{n!}{2\pi i} \int_C \frac{e^{z^2}}{(z-x)^{n+1}} dz \\ &= \frac{1}{(4n)^k} \sum_{l=k}^{n+k} 2^l \left(\sum_{m=0}^k \binom{k}{m} \binom{n}{l-k} H_{n-l+k}(m) \right) G_l^{(k)}(x), \end{aligned}$$

where C is a loop which starts at $-\infty$, encircles the origin once in the positive direction, and the returns $-\infty$.

Remark 1. By using Theorem 1, we can discover many interesting identities related to special polynomials in terms of Genocchi polynomials of higher order.

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