On the K-ring of the classifying space of the generalized quaternion group

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Abstract: We describe the K-ring of the classifying space of the generalized quaternion group in terms of generators and the minimal set of relations. We also compute the order of the main generator in the truncated rings.

Key words: Topological K-theory, representation theory, generalized quaternion group

1. Introduction

The K-ring of the classifying space BQ_{2^n} of the generalized quaternion group Q_{2^n}, $n \geq 3$, is described classically in [4] and [7]. In this note, we describe these rings in a simpler way, by a minimal set of relations on a minimal set of generators. We also make connections between these computations and those done for the lens spaces.

In particular, we compute the order of the main generator of that ring in its truncations, i.e. the $\widetilde{K}$-order of the main vector bundle over the corresponding spherical forms, in a much shorter way than is done in [7]. The reader may find more about the geometric meaning of these orders and quaternionic spherical forms in [4] and [7].

The description of the K-ring is done from the representation ring of the group Q_{2^n} via the Atiyah–Segal Completion Theorem (ASCT), which says that the K-ring of the classifying space of a group is the completion of the representation ring of this group at its augmentation ideal.

Most importantly, we also check the minimality of the relations we found, through the Atiyah–Hirzebruch Spectral Sequence (AHSS), and this will also guarantee that the required completion of the representation ring mentioned in ASCT is thus achieved. The reader who wants to search deep down for the mysterious ASCT and AHSS may start with the papers by their creators, [1], [2].

In connection with and parallel to this problem, the reader should also look at the descriptions for cyclic and dihedral groups. A quick survey for the K-rings of the classifying spaces of cyclic and dihedral groups can be found in [6]. The complete result for the dihedral groups is also published before this paper, in [5], which surprisingly uses the results of this paper for the complicated even case of its problem.

2. Representations

The quaternion group Q_{2^n}, where $n \geq 3$, is generated by 2 elements x and y with the relations $x^{2^{n-1}} = 1$, $x^{2^{n-2}} = y^2$ and $xyx = y$. Note that x generates a cyclic group of order 2^{n-1}, and y generates a cyclic group

of order 4. Note also that $x^{2^{n-2}}$, which is equal to y^2, generates a cyclic group of order 2. We keep in mind these natural group inclusions of the cyclic groups in Q_{2^n}.

There are 4 one-dimensional irreducible complex representations of Q_{2^n}. We will denote them by $1, \eta_1, \eta_2$, and η_3. They are explained by 1×1 matrices in [7].

There are $2^{n-2} - 1$ two-dimensional irreducible complex representations of Q_{2^n}. We will denote them by d_i where $1 \le i \le 2^{n-2} - 1$. Actually, d_i makes sense for any integer i and this will be clarified below.

Since all we need will be the relations that they can generate, we will not describe these representations by matrices here. The descriptions of these representations by 2×2 matrices are given in [7].

Before presenting the relations let us set $2^n = 2m = 4k$. We have this convention from now on. Note that $k \ge 2$ and it is a power of 2 too.

Now, we will list all possible relations in the representation ring $R(Q_{4k})$. First of all, $\eta_3 = \eta_1 \eta_2$. Since $\eta_1^2 = \eta_2^2 = 1$, we also have the relation $\eta_3^2 = 1$. For d_i's, we have the start $d_0 = 1 + \eta_1$ and we have the end $d_k = \eta_2 + \eta_3$.

The main relation, which is the most important of all, is

$$d_i d_j = d_{i+j} + d_{i-j}.$$

This relation makes sense for any integer couple i, j because of the following fact: $d_i = d_{m-i}$ for all integers i.

Another set of relations are for the products of the one- and two-dimensional representations, and they are: $\eta_1 d_i = d_i$ and $\eta_2 d_i = d_{k-i}$ for all i. Since, $\eta_3 = \eta_1 \eta_2$, it follows that $\eta_3 d_i = d_{k-i}$ for all i, the same as η_2.

We deduce from the relations above that the representation ring of Q_{2^n} is just generated by η_1, η_2, and d_1, by means of tensor products and direct sums. The minimal polynomials on η_1, η_2, and d_1 that define the ring can be found from these relations, but we will do that in our new variables.

3. Cohomology

Integral cohomology of Q_{4k}, $k \ge 2$, is the following and can be found in [3]:

$$H^p(BQ_{4k}; Z) = \begin{cases} Z & p = 0 \\ Z_2 \oplus Z_2 & \text{if} \quad p = 4s + 2 \\ Z_{4k} & \text{if} \quad p = 4s,\ s \ge 1 \\ 0 & \text{if} \quad p \text{ is odd.} \end{cases}$$

Note that the odd dimensional cohomology vanishes. Because of that, the AHSS, which converges to $K(BQ_{4k})$, collapses on page 2 so that the K-ring is completely determined by the integral cohomology and vice versa. Here, we also notice that the cohomology is periodic.

The relations of the cohomology ring can also be found in [3]. Note that these relations are quite different than those in the K-ring. We do not try to find connections between these relations. We just compare the orders of the elements of $K(BQ_{4k})$ in the filtrations of the spectral sequence with the sizes of the cohomology groups to prove that the relations we found are minimal.

4. K-Rings

Corresponding to the representations η_1, η_2, and d_1, there are induced vector bundles over the classifying space BQ_{4k} and we denote them by the same letters. We will set the reduced vector bundles as $v_1 = \eta_1 - 1$, $v_2 = \eta_2 - 1$, and, most importantly, the main element $\phi = d_1 - 2$. Due to the ASCT, the elements v_1, v_2, and ϕ generate $K(BQ_{4k})$. All we need is to find the minimal relations on these generators so that the ring is well described.

First of all, since $\eta_1^2 = \eta_2^2 = 1$, we have the following relations:

$$v_1^2 = -2v \text{ and } v_2^2 = -2v_2 \text{ (\textbf{Relations 1 \& 2})}.$$

We note that the above relations are the standard relations for real line bundles over the classifying spaces. These small and simple relations on v_i's explain the cohomology groups $H^{4s+2}(BQ_{4k}; Z) = Z_2 \oplus Z_2$, which are the $(4s+2)$-th filtrations $E_\infty^{4s+2, -4s-2}$ on the main diagonal of the AHSS. The first Z_2 in the direct sum is generated by v_1^s and the second Z_2 is generated by v_2^s where $s \geq 1$.

Next we should explain Z_{4k}'s that occur in the cohomology ring; in other words, we should explain the filtrations $E_\infty^{4s, -4s}$, $s \geq 1$, of the AHSS. This will not be easy.

Recall the natural inclusion of Z_{2k} in Q_{4k} defined by the element $x \in Q_{4k}$. This inclusion results in a natural ring homomorphism $K(BQ_{4k}) \to K(BZ_{2k})$.

Under this homomorphism, the image of the virtual bundle $d_i - 2$ in $K(BZ_{2k})$ is $\eta^i + \eta^{-i} - 2$, where η is the one and only generator of $K(BZ_{2k})$ and the one and only relation it satisfies is $\eta^{2k} = 1$. We set $w = \eta + \eta^{-1} - 2$ in $K(BZ_{2k})$. The element w generates a subring of $K(BZ_{2k})$ that is isomorphic to the subring of $KO(BZ_{2k})$, solely generated by w. Actually it is almost isomorphic, except for a Z_2 direct summand generated by the tautological one-dimensional reduced real bundle, traditionally denoted by λ. Note that $KO(BZ_{2k})$ is the real topological K-theory of the space BZ_{2k}.

Hence, under the natural homomorphism mentioned above, the image of $d_i - 2$ in $K(BZ_{2k})$ is $\psi^i(w)$, where ψ^i is the Adams operation of degree i. Let us recall from [6] the effect of (the real) Adams operation of degree i, on the main generator w of $KO(BZ_{2k})$:

$$\psi^i(w) = \sum_{j=1}^{i} \frac{\binom{i}{j}\binom{i+j-1}{j}}{\binom{2j-1}{j}} w^j$$

We name the above polynomial "quadratic binomial of degree i" because of its connection to the real part of a root of unity and because its coefficients are polynomials of i^2. In particular, under the above ring homomorphism, ϕ maps exactly on w.

On the other hand, in the ring $K(BQ_{4k})$, we have $d_{k+1} - d_{k-1} = 0$. Practically, we observe that this gives a polynomial in ϕ of degree $k+1$, with no one-dimensional bundles involved. This is true for any d_i where i is odd. They can be written as a polynomial of ϕ and only ϕ.

In the ring $KO(BZ_{2k})$, we know that the main relation is $\psi^{k+1}(w) - \psi^{k-1}(w) = 0$, [6]. Furthermore, we deduce that $\psi^{k+1}(\phi) - \psi^{k-1}(\phi) = 0$ in the ring $K(BQ_{4k})$ too. We also conclude that

$$d_i - 2 = \psi^i(\phi) \text{ when } i \text{ is odd.}$$

We can also prove this very important fact from the relations of the representation ring without referring to lens spaces, but this would take longer. Lens spaces make this tricky.

The polynomials $g_{2k}(\phi) = \psi^{k+1}(\phi) - \psi^{k-1}(\phi)$ are given by the following series:

$$g_{2k}(\phi) = 4k\phi + \sum_{j=2}^{k} \frac{2k^2 + j - 1}{(j-1)(2j-1)} \binom{k+j-2}{2j-3} \phi^j + \phi^{k+1}$$

Hence, the following relation is satisfied in $K(BQ_{4k})$:

$$g_{2k}(\phi) = 0 \ (\textbf{Relation 3}).$$

From this relation, we deduce that ϕ satisfies a relation in the form

$$4k\phi = f(\phi)\phi^2$$

where $f(\phi) \in K(BQ_{4k})$ is a virtual bundle generated by ϕ. This explains the fact that the 4-th filtration $E_\infty^{4,-4}$ on the last page of the main diagonal of AHSS is generated by maybe ϕ (or maybe $\phi - v_1 - v_2$ etc.) and is isomorphic to $H^4(BQ_{4k}; Z) = Z_{4k}$. We do not want to speculate much about that filtration in the spectral sequence.

By multiplying this relation by powers of ϕ, all $4s$-th filtrations on the main diagonal of AHSS, i.e. all groups $H^{4s}(BQ_{4k}; Z) = Z_{4k}$ in the cohomology, are similarly explained.

However, we are still not done! It turns out that Relation 3 is not minimal. We will prove it when we talk about the minimal relation for the cross product $v_1 v_2$. We also did not explain the products $v_1\phi$ and $v_1\phi$. Without these relations, the ring cannot be completely described, although more or less the filtrations of the diagonal of the AHSS are explained.

Let us first find the minimal relations for the products $v_i\phi$ where $i = 1$ or 2, and explain why they are not needed to occupy any place on the AHSS.

From the relations $\eta_1 d_1 = d_1$, it immediately follows that

$$v_1\phi = -2v_1 \ (\textbf{Relation 4}).$$

This takes care of the product $v_1\phi$. Next we will take care of the product $v_2\phi$. From the relation, $\eta_2 d_1 = d_{k-1}$, since $d_{k-1} = \psi^{k-1}(\phi) + 2$, we obtain

$$v_2\phi = \psi^{k-1}(\phi) - \phi - 2v_2 \ (\textbf{Relation 5}).$$

Therefore, $v_i\phi$ where $i = 1$ or 2 are dependent variables, and we do not have to search a place on the AHSS for them.

Finally let us explain what remained, in other words, let us find the minimal relation for the cross product $v_1 v_2$. It turns out that the main relation is not Relation 3, but that one. In fact, we will discard our favorite relation, Relation 3, from the minimal set of relations.

We will separate the cases $n = 3$ and $n \geq 4$, since it turns out that Q_8 is a little different than the bigger generalized quaternion groups.

For $k = 2$, from the relation $d_1^2 = d_2 + d_0 = 1 + \eta_1 + \eta_2 + \eta_3$, we have

$$v_1 v_2 = 4\phi + \phi^2 - 2v_1 - 2v_2 \ (\textbf{Relation 6, for } n = 3).$$

This is the main relation for $K(BQ_8)$. If we multiply this equation by $\phi + 2$, an amazing thing happens and we find Relation 3. In other words, we can remove Relation 3 from the minimal list of relations that describes the ring.

For $n \geq 4$, starting from the relation $d_1^2 = d_2 + d_0$, by using the relation $d_i^2 = d_{2i} + d_0$ repeatedly, one inside the other, we can obtain the relation $d_k - d_0 = \psi^k(\phi)$. Along the way, amusingly, we obtain the polynomials $\psi^i(\phi)$, where i is a power of 2, in terms of $\psi^{\frac{i}{2}}(\phi)$. Thus, we have

$$v_1 v_2 = \psi^k(\phi) - 2v_2 \text{ (\textbf{Relation 6, for } } n \geq 4).$$

We will throw away Relation 3 for $n \geq 4$ too. We multiply Relation 6 above by $\phi + 2$ and then we use Relations 4 and 5 properly in the equation we obtained, and again amazingly find Relation 3. This also removes doubts from the obscure explanations given above when we derived this relation trickily from the lens spaces.

We sum up everything in:

Theorem 1 $K(BQ_{2^n})$ *is generated by* v_1, v_2, *and* ϕ *with the minimal set of relations (1),(2),(4),(5), and (6) above.*

5. Orders

In [7], Proposition 5.1, the order of the element ϕ in the truncated ring $R(Q_{4k})/\phi^2 R(Q_{4k})$, after a lot of work by huge matrices, is found as $4k$. He used these orders to answer some geometric problems, namely problems about immersion of spaces S^{4N+3}/Q_{4k} called quaternionic spherical forms, in real Euclidean spaces. However, from the relation $4k\phi = f(\phi)\phi^2$ we found above, this is evident.

Similarly, we can find the order of ϕ in the truncated ring $R(Q_{4k})/\phi^{N+1} R(Q_{4k})$ by careful counting. In Relation 3, we observe that the coefficient of ϕ^2 is $\frac{k(2k^2+1)}{3}$ whose primary 2 factor is $k = 2^{n-2}$. Since the coefficient of ϕ is $4k = 2^n$, the jump between them is $\frac{2^n}{2^{n-2}} = 4$. Therefore, the total count up to ϕ^{N+1} must be $4k.4^{N-1}$. Hence, we have

Corollary 2 *The order of* ϕ *in* $K(S^{4N+3}/Q_{4k})$ *is* 2^{n+2N-2}.

References

[1] Atiyah MF, Hirzebruch F. Vector bundles and homogenous spaces. Proc. Symposium in Pure Maths, Amer. Math. Soc. 1961; 3: 7–38.

[2] Atiyah MF, Segal GB. Equivariant K-theory and completion. Journal of Differential Geometry 1969; 3 no. 1-2: 1–18.

[3] Hayami T, Sanada K. Cohomology ring of the generalized quaternion group with coefficients in an order. Communications in Algebra 2002; 30 no. 8: 3611–3628.

[4] Kenso F. J-groups of the quaternionic spherical space forms. Hiroshima Math. J. 1994; 24: 365–399.

[5] Kırdar M. On the K-ring of the classifying space of the dihedral group. International Journal of Pure and Applied Mathematics 2013; 89 no. 3: 357–364.

[6] Kırdar M. Topological K-theory of the classifying spaces of cyclic and dihedral groups. Hacettepe Journal of Mathematics and Statistics 2013; 42 no. 4: 381–385.

[7] Pitt D. Free actions of the generalized quaternion groups on spheres. Proc. London Math. Soc. 1973; (3) 26: 1–18.