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Research Article

Asymptotic Behaviour of Eigenvalues and Eigenfunctions of a Sturm-Liouville Problem with Retarded Argument

Erdoğan Şen,[1,2] Jong Jin Seo,[3] and Serkan Araci[4]

[1] *Department of Mathematics, Faculty of Arts and Science, Namik Kemal University, 59030 Tekirdağ, Turkey*
[2] *Department of Mathematics Engineering, Istanbul Technical University, Maslak, 34469 Istanbul, Turkey*
[3] *Department of Applied Mathematics, Pukyong National University, Busan 608-737, Republic of Korea*
[4] *Department of Mathematics, Faculty of Science and Arts, University of Gaziantep, 27310 Gaziantep, Turkey*

Correspondence should be addressed to Jong Jin Seo; seo2011@pknu.ac.kr

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In the present paper, a discontinuous boundary-value problem with retarded argument at the two points of discontinuities is investigated. We obtained asymptotic formulas for the eigenvalues and eigenfunctions. This is the first work containing two discontinuities points in the theory of differential equations with retarded argument. In that special case the transmission coefficients $\delta = \gamma = 1$ and retarded argument $\Delta \equiv 0$ in the results obtained in this work coincide with corresponding results in the classical Sturm-Liouville operator.

1. Introduction

Delay differential equations arise in many areas of mathematical modelling, for example, population dynamics (taking into account the gestation times), infectious diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modelling, for example, the body's reaction to CO_2, and so forth, in circulating blood) and chemical kinetics (such as mixing reactants), the navigational control of ships and aircraft, and more general control problems. Also, differential equations and nonlinear differential equations have been studied by many mathematicians in several ways for a long time cf. [1–20].

Boundary value problems for differential equations of the second order with retarded argument were studied in [5–10, 13–16], and various physical applications of such problems can be found in [6].

In the papers [13–16], the asymptotic formulas for the eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument and a spectral parameter in the boundary conditions were derived. In spite of their being already a long years, these subjects are still today enveloped in an aura of mystery within scientific community although they have penetrated numerous mathematical field.

The asymptotic formulas for the eigenvalues and eigenfunctions of the Sturm-Liouville problem with the spectral parameter in the boundary condition were obtained in [17–20].

In this paper we study the eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument. Namely, we consider the boundary value problem for the differential equation

$$y''(x) + q(x)\, y(x - \Delta(x)) + \lambda y(x) = 0, \tag{1}$$

on $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$, with boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \tag{3}$$

and transmission conditions

$$y(h_1-0)-\delta y(h_1+0)=0, \tag{4}$$

$$y'(h_1-0)-\delta y'(h_1+0)=0, \tag{5}$$

$$y(h_2-0)-\gamma y(h_2+0)=0, \tag{6}$$

$$y'(h_2-0)-\gamma y'(h_2+0)=0, \tag{7}$$

where the real-valued function $q(x)$ is continuous in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ and has finite limits

$$q(h_1\pm 0)=\lim_{x\to h_1\pm 0}q(x), \qquad q(h_2\pm 0)=\lim_{x\to h_2\pm 0}q(x), \tag{8}$$

the real-valued function $\Delta(x)\ge 0$ continuous in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ and has finite limits

$$\Delta(h_1\pm 0)=\lim_{x\to h_1\pm 0}\Delta(x), \qquad \Delta(h_2\pm 0)=\lim_{x\to h_2\pm 0}\Delta(x); \tag{9}$$

if $x\in[0,h_1)$ then $x-\Delta(x)\ge 0$; if $x\in(h_1,h_2)$ then $x-\Delta(x)\ge h_1$; if $x\in(h_2,\pi)$ then $x-\Delta(x)\ge h_2$; λ is a real spectral parameter; $h_1,h_2,\alpha,\beta,\delta,\gamma\neq 0$ are arbitrary real numbers such that $0<h_1<h_2<\pi$ and $\sin\alpha\sin\beta\neq 0$.

It must be noted that some problems with transmission conditions which arise in mechanics (thermal condition problem for a thin laminated plate) were studied in [20].

Let $w_1(x,\lambda)$ be a solution of (1) on $[0,h_1]$, satisfying the initial conditions

$$w_1(0,\lambda)=\sin\alpha, \qquad w_1'(0,\lambda)=-\cos\alpha. \tag{10}$$

The conditions (10) define a unique solution of (1) on $[0,h_1]$ ([5], page 12).

After defining the above solution, then we will define the solution $w_2(x,\lambda)$ of (1) on $[h_1,h_2]$ by means of the solution $w_1(x,\lambda)$ using the initial conditions

$$w_2(h_1,\lambda)=\delta^{-1}w_1(h_1,\lambda), \qquad w_2'(h_1,\lambda)=\delta^{-1}w_1'(h_1,\lambda). \tag{11}$$

The conditions (11) define a unique solution of (1) on $[h_1,h_2]$.

After describing the above solution, then we will give the solution $w_3(x,\lambda)$ of (1) on $[h_2,\pi]$ by means of the solution $w_2(x,\lambda)$ using the initial conditions

$$w_3(h_2,\lambda)=\gamma^{-1}w_2(h_2,\lambda), \qquad w_3'(h_2,\lambda)=\gamma^{-1}w_2'(h_2,\lambda). \tag{12}$$

The conditions (12) define a unique solution of (1) on $[h_2,\pi]$.

Consequently, the function $w(x,\lambda)$ is defined on $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ by the equality

$$w(x,\lambda)=\begin{cases} w_1(x,\lambda), & x\in[0,h_1),\\ w_2(x,\lambda), & x\in(h_1,h_2),\\ w_3(x,\lambda), & x\in(h_2,\pi],\end{cases} \tag{13}$$

is a solution of (1) on $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$, which satisfies one of the boundary conditions and four transmission conditions.

Lemma 1. *Let $w(x,\lambda)$ be a solution of* (1) *and $\lambda>0$. Then the following integral equations hold:*

$$w_1(x,\lambda)=\sin\alpha\cos sx-\frac{\cos\alpha}{s}\sin sx-\frac{1}{s}\int_0^x q(\tau)\sin s(x-\tau)\,w_1(\tau-\Delta(\tau),\lambda)\,d\tau \quad \left(s=\sqrt{\lambda},\ \lambda>0\right), \tag{14}$$

$$w_2(x,\lambda)=\frac{1}{\delta}w_1(h_1,\lambda)\cos s(x-h_1)+\frac{w_1'(h_1,\lambda)}{s\delta}\sin s(x-h_1)-\frac{1}{s}\int_{h_1}^x q(\tau)\sin s(x-\tau)\,w_2(\tau-\Delta(\tau),\lambda)\,d\tau \quad \left(s=\sqrt{\lambda},\lambda>0\right), \tag{15}$$

$$w_3(x,\lambda)=\frac{1}{\gamma}w_2(h_2,\lambda)\cos s(x-h_2)+\frac{w_2'(h_2,\lambda)}{s\gamma}\sin s(x-h_2)-\frac{1}{s}\int_{h_2}^x q(\tau)\sin s(x-\tau)\,w_3(\tau-\Delta(\tau),\lambda)\,d\tau \quad \left(s=\sqrt{\lambda},\lambda>0\right). \tag{16}$$

Proof. To prove this lemma, it is enough to substitute $-s^2w_1(\tau,\lambda)-w_1''(\tau,\lambda)$, $-s^2w_2(\tau,\lambda)-w_2''(\tau,\lambda)$, and $-s^2w_3(\tau,\lambda)-w_3''(\tau,\lambda)$ instead of $-q(\tau)w_1(\tau-\Delta(\tau),\lambda)$, $-q(\tau)w_2(\tau-\Delta(\tau),\lambda)$, and $-q(\tau)w_3(\tau-\Delta(\tau),\lambda)$ in the integrals in (14), (15), and (16), respectively, and integrate by parts twice. ☐

Theorem 2. *The problem* (1)–(7) *can have only simple eigenvalues.*

Proof. Let $\tilde{\lambda}$ be an eigenvalue of the problem (1)–(7) and

$$\tilde{y}(x,\tilde{\lambda})=\begin{cases}\tilde{y}_1(x,\tilde{\lambda}), & x\in[0,h_1),\\ \tilde{y}_2(x,\tilde{\lambda}), & x\in(h_1,h_2),\\ \tilde{y}_3(x,\tilde{\lambda}), & x\in(h_2,\pi],\end{cases} \tag{17}$$

be a corresponding eigenfunction. Then, from (2) and (10), it follows that the determinant

$$W\left[\tilde{y}_1(0,\tilde{\lambda}),w_1(0,\tilde{\lambda})\right]=\begin{vmatrix}\tilde{y}_1(0,\tilde{\lambda}) & \sin\alpha\\ \tilde{y}_1'(0,\tilde{\lambda}) & -\cos\alpha\end{vmatrix}=0, \tag{18}$$

and, by Theorem 2.2.2, in [5] the functions $\tilde{y}_1(x,\tilde{\lambda})$ and $w_1(x,\tilde{\lambda})$ are linearly dependent on $[0,h_1]$. We can also prove that the functions $\tilde{y}_2(x,\tilde{\lambda})$ and $w_2(x,\tilde{\lambda})$ are linearly dependent on

$[h_1, h_2]$ and $\widetilde{y}_3(x,\widetilde{\lambda})$ and $w_3(x,\widetilde{\lambda})$ are linearly dependent on $[h_2,\pi]$. Hence

$$\widetilde{y}_i\left(x,\widetilde{\lambda}\right) = K_i w_i\left(x,\widetilde{\lambda}\right) \quad (i=1,2,3), \tag{19}$$

for some $K_1 \neq 0$, $K_2 \neq 0$, and $K_3 \neq 0$. We must show that $K_1 = K_2$ and $K_2 = K_3$. Suppose that $K_2 \neq K_3$. From the equalities (6) and (19), we have

$$\begin{aligned}
&\widetilde{y}\left(h_2-0,\widetilde{\lambda}\right) - \gamma\widetilde{y}\left(h_2+0,\widetilde{\lambda}\right) \\
&\quad = \widetilde{y_2}\left(h_2,\widetilde{\lambda}\right) - \gamma\widetilde{y_3}\left(h_2,\widetilde{\lambda}\right) \\
&\quad = K_2 w_2\left(h_2,\widetilde{\lambda}\right) - \gamma K_3 w_3\left(h_2,\widetilde{\lambda}\right) \\
&\quad = K_2 \gamma w_3\left(h_2,\widetilde{\lambda}\right) - K_3 \gamma w_3\left(h_2,\widetilde{\lambda}\right) \\
&\quad = \gamma\left(K_2-K_3\right) w_3\left(h_2,\widetilde{\lambda}\right) = 0.
\end{aligned} \tag{20}$$

Since $\gamma(K_2-K_3) \neq 0$ it follows that

$$w_3\left(h_2,\widetilde{\lambda}\right) = 0. \tag{21}$$

By the same procedure from equality (7) we can derive that

$$w_3'\left(h_2,\widetilde{\lambda}\right) = 0. \tag{22}$$

From the fact that $w_3(x,\widetilde{\lambda})$ is a solution of the differential (1) on $[h_2,\pi]$ and satisfies the initial conditions (21) and (22), it follows that $w_3(x,\widetilde{\lambda}) = 0$ identically on $[h_2,\pi]$.

By using this method, we may also find

$$\begin{aligned}
w_2\left(h_2,\widetilde{\lambda}\right) &= w_2'\left(h_2,\widetilde{\lambda}\right) = 0, \\
w_1\left(h_1,\widetilde{\lambda}\right) &= w_1'\left(h_1,\widetilde{\lambda}\right) = 0.
\end{aligned} \tag{23}$$

From the latter discussions of $w_3(x,\widetilde{\lambda})$, it follows that $w_2(x,\widetilde{\lambda}) = 0$ and $w_1(x,\widetilde{\lambda}) = 0$ identically on (h_1,h_2) and $[0,h_1)$, but this contradicts (10), thus completing the proof. □

2. An Existence Theorem

The function $w(x,\lambda)$ defined in Section 1 is a nontrivial solution of (1) satisfying conditions (2) and (4)–(7). Putting $w(x,\lambda)$ into (3), we get the characteristic equation

$$F(\lambda) \equiv w(\pi,\lambda)\cos\beta + w'(\pi,\lambda)\sin\beta = 0. \tag{24}$$

By Theorem 2 the set of eigenvalues of boundary-value problem (1)–(7) coincides with the set of real roots of (24). Let

$$\begin{gathered}
q_1 = \int_0^{h_1} \left|q(\tau)\right| d\tau, \qquad q_2 = \int_{h_1}^{h_2} \left|q(\tau)\right| d\tau, \\
q_3 = \int_{h_2}^{\pi} \left|q(\tau)\right| d\tau.
\end{gathered} \tag{25}$$

Lemma 3. *(1) Let $\lambda \geq 4q_1^2$. Then for the solution $w_1(x,\lambda)$ of* (14), *the following inequality holds:*

$$\left|w_1(x,\lambda)\right| \leq \frac{1}{q_1}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}, \quad x \in \left[0,h_1\right]. \tag{26}$$

(2) Let $\lambda \geq \max\{4q_1^2, 4q_2^2\}$. Then for the solution $w_2(x,\lambda)$ of (15), *the following inequality holds:*

$$\left|w_2(x,\lambda)\right| \leq \frac{4}{q_1\left|\delta\right|}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}, \quad x \in \left[h_1,h_2\right]. \tag{27}$$

(3) Let $\lambda \geq \max\{4q_1^2, 4q_2^2, 4q_3^2\}$. Then for the solution $w_3(x,\lambda)$ of (16), *the following inequality holds:*

$$\left|w_3(x,\lambda)\right| \leq \frac{16}{q_1\left|\delta\gamma\right|}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}, \quad x \in \left[h_2,\pi\right]. \tag{28}$$

Proof. Let $B_{1\lambda} = \max_{[0,h_1]}|w_1(x,\lambda)|$. Then from (14), it follows that, for every $\lambda > 0$, the following inequality holds:

$$B_{1\lambda} \leq \sqrt{\sin^2\alpha + \frac{\cos^2\alpha}{s^2}} + \frac{1}{s}B_{1\lambda}q_1. \tag{29}$$

If $s \geq 2q_1$ we get (26). Differentiating (14) with respect to x, we have

$$\begin{aligned}
w_1'(x,\lambda) = {} & -s\sin\alpha\,\sin sx - \cos\alpha\,\cos sx \\
& - \int_0^x q(\tau)\cos s(x-\tau)\, w_1(\tau-\Delta(\tau),\lambda)\, d\tau.
\end{aligned} \tag{30}$$

From expressions of (30) and (26), it follows that, for $s \geq 2q_1$, the following inequality holds:

$$\frac{\left|w_1'(x,\lambda)\right|}{s} \leq \frac{1}{q_1}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}. \tag{31}$$

Let $B_{2\lambda} = \max_{[h_1,h_2]}|w_2(x,\lambda)|$. Then from (11), (26), and (31) it follows that, for $s \geq 2q_1$ and $s \geq 2q_2$, the following inequality holds:

$$B_{2\lambda} \leq \frac{2}{q_1\left|\delta\right|}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha} + \frac{1}{2q_2}B_{2\lambda}q_2. \tag{32}$$

Hence, if $\lambda \geq \max\{4q_1^2, 4q_2^2\}$, it reduces to (27). Differentiating (15) with respect to, we get

$$\begin{aligned}
w_2'(x,\lambda) = {} & -\frac{s}{\delta}w_1\left(h_1,\lambda\right)\sin s\left(x-h_1\right) \\
& + \frac{w_1'\left(h_1,\lambda\right)}{\delta}\cos s\left(x-h_1\right) \\
& - \int_{h_1}^x q(\tau)\cos s(x-\tau)\, w_2(\tau-\Delta(\tau),\lambda)\, d\tau. \\
& \left(s=\sqrt{\lambda}, \lambda>0\right).
\end{aligned} \tag{33}$$

From (26) and (33), it follows that, for $s \geq 2q_1$ and $s \geq 2q_2$, the following inequality holds:

$$\frac{\left|w_2'(x,\lambda)\right|}{s} \leq \frac{4}{|\delta|\, q_1}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha}. \qquad (34)$$

Let $B_{3\lambda} = \max_{[h_2,\pi]}|w_3(x,\lambda)|$. Then from (16), (27), and (34) it follows that, for $s \geq 2q_1$, $s \geq 2q_2$ and $s \geq 2q_3$, the following inequality holds:

$$B_{3\lambda} \leq \frac{8}{q_1|\delta\gamma|}\sqrt{4q_1^2\sin^2\alpha+\cos^2\alpha}+\frac{1}{s}B_{3\lambda}q_3. \qquad (35)$$

Hence, if $\lambda \geq \max\{4q_1^2, 4q_2^2, 4q_3^2\}$ we procure (28). □

Theorem 4. *The problem* (1)–(7) *has an infinite set of positive eigenvalues.*

Proof. Differentiating (16) with respect to x, we readily see that

$$\begin{aligned} w_3'(x,\lambda) = &-\frac{s}{\gamma}w_2(h_2,\lambda)\sin s(x-h_2) \\ &+\frac{w_2'(h_2,\lambda)}{\gamma}\cos s(x-h_2) \\ &-\int_{h_2}^{x} q(\tau)\cos s(x-\tau)\, w_3(\tau-\Delta(\tau),\lambda)\,d\tau. \\ &\left(s=\sqrt{\lambda},\lambda>0\right). \end{aligned} \qquad (36)$$

With the helps of (14), (15), (16), (24), (30), and (36), we have the following:

$$\begin{aligned} &\left[\frac{1}{\gamma}\left\{\frac{1}{\delta}\left(\sin\alpha\cos sh_1-\frac{\cos\alpha}{s}\sin sh_1\right.\right.\right. \\ &\qquad\left.-\frac{1}{s}\int_0^{h_1}q(\tau)\sin s(h_1-\tau)\,w_1(\tau-\Delta(\tau),\lambda)\,d\tau\right) \\ &\quad\times\cos s(h_2-h_1) \\ &\quad-\frac{1}{s\delta}\left(s\sin\alpha\sin sh_1+\cos\alpha\cos sh_1\right. \\ &\qquad\left.+\int_0^{h_1}q(\tau)\cos s(h_1-\tau)\,w_1(\tau-\Delta(\tau),\lambda)\,d\tau\right) \\ &\quad\times\sin s(h_2-h_1) \\ &\quad\left.-\frac{1}{s}\int_{h_1}^{h_2}q(\tau)\sin s(h_2-\tau)\,w_2(\tau-\Delta(\tau),\lambda)\,d\tau\right\} \\ &\times\cos s(\pi-h_2) \\ &+\frac{1}{s\gamma}\left\{-\frac{s}{\delta}\left(\sin\alpha\cos sh_1-\frac{\cos\alpha}{s}\sin sh_1\right.\right. \\ &\qquad-\frac{1}{s}\int_0^{h_1}q(\tau)\sin s(h_1-\tau) \\ &\qquad\left.\times\, w_1(\tau-\Delta(\tau),\lambda)\,d\tau\right) \\ &\quad\times\sin s(h_2-h_1) \\ &\quad+\frac{1}{\delta}\left(-s\sin\alpha\sin sh_1-\cos\alpha\cos sh_1\right. \\ &\qquad-\int_0^{h_1}q(\tau)\cos s(h_1-\tau) \\ &\qquad\left.\times w_1(\tau-\Delta(\tau),\lambda)\,d\tau\right) \\ &\quad\times\cos s(h_2-h_1) \\ &\quad\left.-\int_{h_1}^{h_2}q(\tau)\cos s(h_2-\tau)\,w_2(\tau-\Delta(\tau),\lambda)\,d\tau\right\} \\ &\times\sin s(\pi-h_2) \\ &\left.-\frac{1}{s}\int_{h_2}^{\pi}q(\tau)\sin s(\pi-\tau)\,w_3(\tau-\Delta(\tau),\lambda)\,d\tau\right]\cos\beta \\ &+\left[-\frac{s}{\gamma}\left\{\frac{1}{\delta}\left(\sin\alpha\cos sh_1-\frac{\cos\alpha}{s}\sin sh_1\right.\right.\right. \\ &\qquad-\frac{1}{s}\int_0^{h_1}q(\tau)\sin s(h_1-\tau) \\ &\qquad\left.\times w_1(\tau-\Delta(\tau),\lambda)\,d\tau\right) \\ &\quad\times\cos s(h_2-h_1) \\ &\quad-\frac{1}{s\delta}\left(s\sin\alpha\sin sh_1+\cos\alpha\cos sh_1\right. \\ &\qquad+\int_0^{h_1}q(\tau)\cos s(h_1-\tau) \\ &\qquad\left.\times\, w_1(\tau-\Delta(\tau),\lambda)\,d\tau\right) \\ &\quad\times\sin s(h_2-h_1) \\ &\quad\left.-\frac{1}{s}\int_{h_1}^{h_2}q(\tau)\sin s(h_2-\tau)\,w_2(\tau-\Delta(\tau),\lambda)\right\} \\ &\times\sin s(\pi-h_2) \end{aligned}$$

$$+\frac{1}{\gamma}\left\{-\frac{s}{\delta}\times\left(\sin\alpha\cos sh_1-\frac{\cos\alpha}{s}\sin sh_1\right.\right.$$
$$-\frac{1}{s}\int_0^{h_1}q(\tau)\sin s(h_1-\tau)$$
$$\left.\times w_1(\tau-\Delta(\tau),\lambda)\,d\tau\right)$$
$$\times\sin s(h_2-h_1)$$
$$+\frac{1}{\delta}\left(-s\sin\alpha\sin sh_1-\cos\alpha\cos sh_1\right.$$
$$-\int_0^{h_1}q(\tau)\cos s(h_1-\tau)$$
$$\left.\times w_1(\tau-\Delta(\tau),\lambda)\,d\tau\right)$$
$$\times\cos s(h_2-h_1)$$
$$\left.-\int_{h_1}^{h_2}q(\tau)\cos s(h_2-\tau)\,w_2(\tau-\Delta(\tau),\lambda)\,d\tau\right\}$$
$$\times\cos s(\pi-h_2)$$
$$\left.-\int_{h_2}^{\pi}q(\tau)\cos s(\pi-\tau)\,w_3(\tau-\Delta(\tau),\lambda)\,d\tau\right]\sin\beta=0. \tag{37}$$

Let λ be sufficiently big. Then, by (26), (27), and (28), (37) may be rewritten in the following form:

$$-\frac{s\sin\alpha\sin\beta}{\gamma\delta}\left\{\cos sh_2\sin s(\pi-h_2)+\sin sh_2\cos s(\pi-h_2)\right\}+O(1)=0, \tag{38}$$

$$s\sin s\pi+O(1)=0. \tag{39}$$

Obviously, for big s (39) has an infinite set of roots. Thus, the proof of the theorem is completed. □

3. Asymptotic Formulas for Eigenvalues and Eigenfunctions

Now we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following we will assume that is sufficiently big. From (14) and (26), we obtain

$$w_1(x,\lambda)=O(1)\quad\text{on }[0,h_1]. \tag{40}$$

By (15) and (27), this leads to

$$w_2(x,\lambda)=O(1)\quad\text{on }[h_1,h_2]. \tag{41}$$

By (16) and (28), this leads to

$$w_3(x,\lambda)=O(1)\quad\text{on }[h_2,\pi]. \tag{42}$$

The existence and continuity of the derivatives $w'_{1s}(x,\lambda)$ for $0\le x\le h_1$, $|\lambda|<\infty$, $w'_{2s}(x,\lambda)$ for $h_1\le x\le h_2$, $|\lambda|<\infty$ and $w'_{3s}(x,\lambda)$ for $h_2\le x\le\pi$, $|\lambda|<\infty$ follows from Theorem 1.4.1 in [5].

Lemma 5. *The following holds true:*

$$w'_{1s}(x,\lambda)=O(1),\quad x\in[0,h_1], \tag{43}$$
$$w'_{2s}(x,\lambda)=O(1),\quad x\in[h_1,h_2], \tag{44}$$
$$w'_{3s}(x,\lambda)=O(1),\quad x\in[h_2,\pi]. \tag{45}$$

Proof. By differentiating (16) with respect to s, we get, by (43) and (44) the following:

$$w'_{3s}(x,\lambda)=-\frac{1}{s}\int_{h_2}^{x}q(\tau)\sin s(x-\tau)\,w'_{3s}(\tau-\Delta(\tau),\lambda)+Z(x,\lambda),\quad(|Z(x,\lambda)|\le Z_0). \tag{46}$$

Let $D_\lambda=\max_{[h_2,\pi]}|w'_{3s}(x,\lambda)|$. Then the existence of D_λ follows from continuity of derivation for $x\in[h_2,\pi]$. From (46)

$$D_\lambda\le\frac{1}{s}q_3D_\lambda+Z_0. \tag{47}$$

Now let $s\ge 2q_3$. Then $D_\lambda\le 2Z_0$ and the validity of the asymptotic formula (45) follows. Formulas (43) and (44) may be proved analogically. □

Theorem 6. *Let n be a natural number. For each sufficiently big n there is exactly one eigenvalue of the problem* (1)–(7) *near n^2.*

Proof. We consider the expression which is denoted by $O(1)$ in (39). If formulas (40)–(45) are taken into consideration, it can be shown by differentiation with respect to s that for big s this expression has bounded derivative. We will show that, for big n, only one root (39) lies near to each n. We consider the function $\phi(s)=s\sin s\pi+O(1)$. Its derivative, which has the form $\phi'(s)=\sin s\pi+s\pi\cos s\pi+O(1)$, does not vanish for s close to n for sufficiently big n. Thus our assertion follows by Rolle's Theorem. □

Let n be sufficiently big. In what follows we will denote by $\lambda_n=s_n^2$ the eigenvalue of the problem (1)–(7) situated near n^2. We set $s_n=n+\delta_n$. Then from (39) it follows that $\delta_n=O(1/n)$. Consequently

$$s_n=n+O\left(\frac{1}{n}\right), \tag{48}$$
$$\lambda_n=n^2+O(1). \tag{49}$$

Formula (48) makes it possible to obtain asymptotic expressions for eigenfunction of the problem (1)–(7). From (14), (30), and (40), we get

$$w_1(x,\lambda)=\sin\alpha\cos sx+O\left(\frac{1}{s}\right), \tag{50}$$
$$w'_1(x,\lambda)=-s\sin\alpha\sin sx+O(1). \tag{51}$$

From expressions of (15), (31), (39), and (41), we easily see that

$$w_2(x,\lambda) = \frac{\sin\alpha}{\delta}\cos sx + O\left(\frac{1}{s}\right),$$
$$w_3(x,\lambda) = \frac{\sin\alpha}{\delta\gamma}\cos sx + O\left(\frac{1}{s}\right). \tag{52}$$

By substituting (48) into (50), (52), we find that

$$u_{1n} = w_1(x,\lambda_n) = \sin\alpha\cos nx + O\left(\frac{1}{n}\right),$$
$$u_{2n} = w_2(x,\lambda_n) = \frac{\sin\alpha}{\delta}\cos nx + O\left(\frac{1}{n}\right), \tag{53}$$
$$u_{3n} = w_3(x,\lambda_n) = \frac{\sin\alpha}{\delta\gamma}\cos nx + O\left(\frac{1}{n}\right).$$

Hence the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x) = \begin{cases} \sin\alpha\cos nx + O\left(\frac{1}{n}\right), & \text{for } x \in [0,h_1), \\ \frac{\sin\alpha}{\delta}\cos nx + O\left(\frac{1}{n}\right), & \text{for } x \in (h_1,h_2), \\ \frac{\sin\alpha}{\delta\gamma}\cos nx + O\left(\frac{1}{n}\right), & \text{for } x \in (h_2,\pi]. \end{cases} \tag{54}$$

Under some additional conditions the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled.

(a) The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ and have finite limits $q'(h_1\pm0) = \lim_{x\to h_1\pm0} q'(x)$, $q'(h_2\pm0) = \lim_{x\to h_2\pm0} q'(x)$ and $\Delta''(h_1\pm0) = \lim_{x\to h_1\pm0}\Delta''(x)$, $\Delta''(h_2\pm0) = \lim_{x\to h_2\pm0}\Delta''(x)$, respectively.

(b) $\Delta'(x) \le 1$ in $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$, $\Delta(0) = 0$, $\lim_{x\to h_1+0}\Delta(x) = 0$, and $\lim_{x\to h_2+0}\Delta(x) = 0$.

It is easy to see that, using (b)

$$x - \Delta(x) \ge 0, \quad x \in [0,h_1),$$
$$x - \Delta(x) \ge h_1, \quad x \in (h_1,h_2), \tag{55}$$
$$x - \Delta(x) \ge h_2, \quad x \in (h_2,\pi]$$

are obtained.

By (50), (52), and (55), we have

$$w_1(\tau-\Delta(\tau),\lambda) = \sin\alpha\cos s(\tau-\Delta(\tau)) + O\left(\frac{1}{s}\right), \tag{56}$$

$$w_2(\tau-\Delta(\tau),\lambda) = \frac{\sin\alpha}{\delta}\cos s(\tau-\Delta(\tau)) + O\left(\frac{1}{s}\right), \tag{57}$$

$$w_3(\tau-\Delta(\tau),\lambda) = \frac{\sin\alpha}{\delta\gamma}\cos s(\tau-\Delta(\tau)) + O\left(\frac{1}{s}\right) \tag{58}$$

on $[0,h_1)$, (h_1,h_2) and $(h_2,\pi]$, respectively.

Under the conditions (a) and (b) the following formulas:

$$\int_0^x q(\tau)\cos s(2\tau-\Delta(\tau))\,d\tau = O\left(\frac{1}{s}\right),$$
$$\int_0^x q(\tau)\sin s(2\tau-\Delta(\tau))\,d\tau = O\left(\frac{1}{s}\right) \tag{59}$$

can be proved by the same technique in Lemma 3.3.3 in [5].

Putting the expressions (56), (57), and (58) into (37), and then using (59), after long operations we have

$$\frac{\cos s\pi\sin(\alpha-\beta) - s\sin\alpha\sin\beta\sin s\pi}{\delta\gamma} - \frac{\sin\alpha\sin\beta}{2\delta\gamma}\int_0^\pi q(\tau)\cos s(\pi-\Delta(\tau))\,d\tau + O\left(\frac{1}{s}\right) = 0. \tag{60}$$

Hence

$$\tan s\pi = \frac{1}{s}\left(\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} - \frac{1}{2}\int_0^\pi q(\tau)\cos s(\pi-\Delta(\tau))\,d\tau\right) + O\left(\frac{1}{s^2}\right). \tag{61}$$

Again, if we take $s_n = n + \delta_n$, then from (48)

$$\tan((n+\delta_n)\pi) = \tan\delta_n\pi = \frac{1}{n}\left(\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} - \frac{1}{2}\int_0^\pi q(\tau)\cos s(\pi-\Delta(\tau))\,d\tau\right) + O\left(\frac{1}{n^2}\right). \tag{62}$$

Hence for big n,

$$\delta_n = \frac{1}{n\pi}\left(\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} - \frac{1}{2}\int_0^\pi q(\tau)\cos s(\pi-\Delta(\tau))\,d\tau\right) + O\left(\frac{1}{n^2}\right) \tag{63}$$

and finally

$$s_n = n + \frac{1}{n\pi}\left(\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} - \frac{1}{2}\int_0^\pi q(\tau)\cos s(\pi-\Delta(\tau))\,d\tau\right) + O\left(\frac{1}{n^2}\right). \tag{64}$$

Thus, we have proven the following theorem.

Theorem 7. *If conditions (a) and (b) are satisfied, then the eigenvalues* $\lambda_n = s_n^2$ *of the problem* (1)–(7) *have the* (64) *asymptotic formula for* $n \to \infty$.

Now, we may obtain sharper asymptotic formulas for the eigenfunctions. From (14), (56), and (59), we have

$$\begin{aligned} w_1(x,\lambda) = {} & \sin\alpha\cos sx\left[1+\frac{1}{2s}\int_0^x q(\tau)\sin s\Delta(\tau)\,d\tau\right] \\ & -\frac{\sin sx}{s}\left[\cos\alpha+\frac{\sin\alpha}{2}\int_0^x q(\tau)\cos s\Delta(\tau)\,d\tau\right] \\ & + O\left(\frac{1}{s^2}\right), \quad x\in[0,h_1). \end{aligned} \tag{65}$$

Now, replacing s by s_n and using (64), we have

$$\begin{aligned} & u_{1n}(x) \\ & = w_1(x,\lambda_n) \\ & = \sin\alpha\left\{\cos nx\left[1+\frac{1}{2n}\int_0^x q(\tau)\sin(n\Delta(\tau))\,d\tau\right]\right. \\ & \qquad -\frac{\sin nx}{n\pi}\left[\left(\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}\right.\right. \\ & \qquad\qquad \left.-\frac{1}{2}\int_0^\pi q(\tau)\cos(n(\pi-\Delta(\tau)))\,d\tau\right)x \\ & \qquad\qquad \left.\left.\left.+\left(\cot\alpha+\frac{1}{2}\int_0^x q(\tau)\cos(n\Delta(\tau))\,d\tau\right)\pi\right]\right]\right\} \\ & \quad + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{66}$$

From (15), (57), (59), and (65) we have

$$\begin{aligned} w_2(x,\lambda) = {} & \frac{\sin\alpha\cos sx}{\delta}\left[1+\frac{1}{2s}\int_0^x q(\tau)\sin s\Delta(\tau)\,d\tau\right] \\ & -\frac{\sin sx}{s\delta}\left[\cos\alpha+\frac{\sin\alpha}{2}\int_0^x q(\tau)\cos s\Delta(\tau)\,d\tau\right] \\ & + O\left(\frac{1}{s^2}\right), \quad x\in(h_1,h_2). \end{aligned} \tag{67}$$

Now, replacing s by s_n and using (64), we have

$$\begin{aligned} & u_{2n}(x) \\ & = w_2(x,\lambda_n) \\ & = \frac{\sin\alpha}{\delta}\left\{\cos nx\left[1+\frac{1}{2n}\int_0^x q(\tau)\sin(n\Delta(\tau))\,d\tau\right]\right. \\ & \qquad -\frac{\sin nx}{n\pi}\left[\left(\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}\right.\right. \\ & \qquad\qquad \left.-\frac{1}{2}\int_0^\pi q(\tau)\cos(n(\pi-\Delta(\tau)))\,d\tau\right)x \\ & \qquad\qquad \left.\left.\left.+\left(\cot\alpha+\frac{1}{2}\int_0^x q(\tau)\cos(n\Delta(\tau))\,d\tau\right)\pi\right]\right]\right\} \\ & \quad + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{68}$$

From (16), (58), (59), (65), and (67) and after long operations, we have

$$\begin{aligned} w_3(x,\lambda) = {} & \frac{\sin\alpha\cos sx}{\delta\gamma}\left[1+\frac{1}{2s}\int_0^x q(\tau)\sin s\Delta(\tau)\,d\tau\right] \\ & -\frac{\sin sx}{s\delta\gamma}\left[\cos\alpha+\frac{\sin\alpha}{2}\int_0^x q(\tau)\cos s\Delta(\tau)\,d\tau\right] \\ & + O\left(\frac{1}{s^2}\right), \quad x\in(h_2,\pi]. \end{aligned} \tag{69}$$

Now replacing s by s_n and using (64), we have

$$\begin{aligned} & u_{3n}(x) \\ & = w_3(x,\lambda_n) \\ & = \frac{\sin\alpha}{\delta\gamma}\left\{\cos nx\left[1+\frac{1}{2n}\int_0^x q(\tau)\sin(n\Delta(\tau))\,d\tau\right]\right. \\ & \qquad -\frac{\sin nx}{n\pi}\left[\left(\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}\right.\right. \\ & \qquad\qquad \left.-\frac{1}{2}\int_0^\pi q(\tau)\cos(n(\pi-\Delta(\tau)))\,d\tau\right)x \\ & \qquad\qquad \left.\left.\left.+\left(\cot\alpha+\frac{1}{2}\int_0^x q(\tau)\cos(n\Delta(\tau))\,d\tau\right)\pi\right]\right]\right\} \\ & \quad + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{70}$$

Thus, we have proven the following theorem.

Theorem 8. *If conditions (a) and (b) are satisfied, then the eigenfunctions* $u_n(x)$ *of the problem* (1)–(7) *have the following asymptotic formula for* $n \to \infty$:

$$u_n(x) = \begin{cases} u_{1n}(x), & x\in[0,h_1), \\ u_{2n}(x), & x\in(h_1,h_2), \\ u_{3n}(x), & x\in(h_2,\pi], \end{cases} \tag{71}$$

where $u_{1n}(x)$, $u_{2n}(x)$, *and* $u_{3n}(x)$ *are determined as in (49),* (68), *and* (70), *respectively.*

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