MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
6 (1) 64-69 (2018) ©MSAEN

Position Vectors of General Helices According to Type-2 Bishop Frame in E^3

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(Communicated by Ákos G. HORVÁTH)

Abstract
In this paper, we study the position vector of a general helix according to type-2 Bishop frame in the 3-dimensional Euclidean space E^3. Moreover we determine the natural representation of a general helix in E^3.

Keywords: *Position vector; type-2 Bishop frame.*

AMS Subject Classification (2010): *53A04.*

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1. Introduction

The theory of curves has an important role in differential geometry. One of the most interesting curve used in nature and science is helix. A helix is a curve with non-vanishing curvature and torsion. This fascinating curve arises in nature such that DNA double, carbon nano-tubes, nano springs, etc.[6, 8]. Intrinsically a helix also known a circular helix or W-curve is a special case of general helix [5, 7]. A general helix or a curve of constant slope is determined that the tangent makes a constant angle with a fixed straight line called the axis of general helix [3]. An important result for general helix stated by Lancret (1802) and proved by de Saint Venant (1945) is: a necessary and sufficient condition that a curve be general helix is that ratio of curvature to torsion be constant [11] .

On the other hand, L.R. Bishop defined Bishop frame, which is known alternative or parallel frame of the curves with the help of parallel vector fields [4]. Then, S. Yilmaz and M. Turgut introduced a new version of the Bishop frame which is called type-2 Bishop frame [12]. Thereafter, E. Ozyilmaz studied classical differential geometry of curves according to type-2 Bishop trihedra [9].

Position vectors of curves investigated by many researchers, for example in [1] and [2] position vectors of general and slant helices in 3-dimensional Euclidean space are researched, in [7] position vector of a spacelike W-curve in Minkowski space is obtained and in [10] position vectors of admissible curves in 3-dimensional pseudo-Galilean space are determined.

In this study we research position vectors of a general helix according to type-2 Bishop frame in the 3-dimensional Euclidean space E^3, and we obtain the natural representation of a general helix.

2. Preliminaries

The standard flat metric of 3-dimensional Euclidean space E^3 is given by

$$\langle\ ,\ \rangle : dx_1^2 + dx_2^2 + dx_3^2 \tag{2.1}$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. For an arbitrary vector x in E^3, the norm of this vector is defined by $\|x\| = \sqrt{\langle x, x\rangle}$. α is called a unit speed curve, if $\langle \alpha', \alpha' \rangle = 1$. Suppose that $\{t, n, b\}$ is the moving

Received : *14–03–2018,* ***Accepted :*** *19–04–2018*

Frenet–Serret frame along the curve α in E^3. For a unit speed curve α, the Frenet-Serret formulae can be given as

$$\begin{aligned} t' &= \kappa n \\ n' &= -\kappa t + \tau b \\ b' &= -\tau n \end{aligned} \tag{2.2}$$

where

$$\langle t,t\rangle = \langle n,n\rangle = \langle b,b\rangle = 1,$$
$$\langle t,n\rangle = \langle t,b\rangle = \langle n,b\rangle = 0.$$

and here, $\kappa = \kappa(s) = \|t'(s)\|$ and $\tau = \tau(s) = -\langle n, b'\rangle$. Furthermore, the torsion of the curve α can be given

$$\tau = \frac{[\alpha', \alpha'', \alpha''']}{\kappa^2}.$$

Along the paper, we assume that $\kappa \neq 0$ and $\tau \neq 0$.

Bishop frame is an alternative approachment to define a moving frame. Assume that $\alpha(s)$ is a unit speed regular curve in E^3. The type-2 Bishop frame of the $\alpha(s)$ is expressed as [12]

$$\begin{aligned} N_1' &= -k_1 B, \\ N_2' &= -k_2 B, \\ B' &= k_1 N_1 + k_2 N_2. \end{aligned} \tag{2.3}$$

The relation matrix may be expressed as

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix} = \begin{bmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}. \tag{2.4}$$

where $\theta(s) = \int_0^s \kappa(s)\,ds$. Then, type-2 Bishop curvatures can be defined in the following

$$k_1(s) = -\tau(s)\cos\theta(s),$$
$$k_2(s) = -\tau(s)\sin\theta(s).$$

On the other hand,

$$\theta' = \kappa = \frac{\left(\frac{k_2}{k_1}\right)'}{1 + \left(\frac{k_2}{k_1}\right)^2}.$$

The frame $\{N_1, N_2, B\}$ is properly oriented, τ and $\theta(s) = \int_0^s \kappa(s)\,ds$ are polar coordinates for the curve α . Then, $\{N_1, N_2, B\}$ is called type-2 Bishop trihedra and k_1, k_2 are called Bishop curvatures.

3. Position Vectors of a General Helix According to type-2 Bishop Frame

Theorem 3.1. *Let $\alpha(x)$ be a general helix with $k_1(x) \neq 0$. Then its position vector with respect to type-2 Bishop frame is given by*

$$\alpha(x) = \lambda N_1 + \mu N_2 + \gamma B \tag{3.1}$$

where

$$\begin{aligned}
\lambda &= \int \frac{1}{k_1(x)}dx \\
&- \int \left[\left(\frac{1}{\sqrt{1+m^2}} \int \cos\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) dx + c_1 \right) \right. \\
&\quad \left. \sin\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) \right] dx \\
&+ \int \left[\left(\frac{1}{\sqrt{1+m^2}} \int \sin\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) dx + c_2 \right) \right. \\
&\quad \left. \cos\left(\sqrt{1+m^2} \int k_1(x) \right) dx \right] dx \\
\mu &= -m \int \left[\left(\frac{1}{\sqrt{1+m^2}} \int \cos\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) dx + c_1 \right) \right. \\
&\quad \left. \sin\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) \right] dx \\
&+ m \int \left[\left(\frac{1}{\sqrt{1+m^2}} \int \sin\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) dx + c_2 \right) \right. \\
&\quad \left. \cos\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) \right] dx \\
\gamma &= \left[\frac{1}{\sqrt{1+m^2}} \int \cos\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) dx + c_1 \right] \sin\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) \\
&- \left[\frac{1}{\sqrt{1+m^2}} \int \sin\left(\sqrt{1+m^2} \int k_1(x)\,dx \right) dx + c_2 \right] \cos\left(\sqrt{1+m^2} \int k_1(x)\,dx \right)
\end{aligned}$$

and where $\frac{k_2(x)}{k_1(x)} = m = constant.$

Proof. Let $\alpha(x)$ be a general helix with $k_1(x) \neq 0$ in E^3. If λ, μ, γ are differentiable functions of $x \in I \subset \mathbb{R}$ then its position vectors according to type-2 Bishop frame can be written

$$\alpha(x) = \lambda(x) N_1(x) + \mu(x) N_2(x) + \gamma(x) B(x). \tag{3.2}$$

If N_1 is taken instead of tangent vector, differentiating above equation according to x and considering type-2 Bishop frame we have the following

$$\begin{aligned}
\lambda' + \gamma k_1 - 1 &= 0 \\
\mu' + \gamma k_2 &= 0 \\
\gamma' - \lambda k_1 - \mu k_2 &= 0.
\end{aligned} \tag{3.3}$$

If we can change the variable x by the variable $\theta = \int k_1(x)\,dx$, then all functions of x will turn into the functions of θ (such that $\lambda(\theta) = (\lambda \circ x)(\theta)$). Therefore the equation (3.3) is written as

$$\dot{\lambda} + \gamma - \frac{1}{k_1} = 0 \tag{3.4}$$

$$\dot{\mu} + \gamma \frac{k_2}{k_1} = 0 \tag{3.5}$$

$$\dot{\gamma} - \lambda - \mu \frac{k_2}{k_1} = 0 \tag{3.6}$$

where the dot denote the derivative according to θ. Then differentiating equation (3.6) and substiuting the equations (3.4) and (3.5) in the equation (3.6) we get the following differential equation

$$\ddot{\gamma} + \left(1 + \frac{k_2^2}{k_1^2} \right) \gamma - \frac{1}{k_1} = 0 \tag{3.7}$$

If we solve above equation and setting $\theta = \int k_1(x)\,dx$ we obtain the equation (3.1) which complete the proof. □

Lemma 3.1. *Let $\alpha(x)$ be a circular helix, then its position vector with respect to type-2 Bishop frame is given by*

$$\alpha(x) = \lambda N_1 + \mu N_2 + \gamma B \tag{3.8}$$

where

$$\begin{aligned}
\lambda &= x - k_1 \int \left[\frac{k_1}{{k_1}^2 + {k_2}^2} + c_1 \cos \sqrt{{k_2}^2 + {k_1}^2}x + c_2 \sin \sqrt{{k_2}^2 + {k_1}^2}x \right] dx \\
\mu &= -\frac{k_2}{k_1} \int \left[\frac{k_1}{{k_1}^2 + {k_2}^2} + c_1 \cos \sqrt{{k_2}^2 + {k_1}^2}x + c_2 \sin \sqrt{{k_2}^2 + {k_1}^2}x \right] dx \\
\gamma &= \frac{k_1}{{k_1}^2 + {k_2}^2} + c_1 \cos \sqrt{{k_2}^2 + {k_1}^2}x + c_2 \sin \sqrt{{k_2}^2 + {k_1}^2}x
\end{aligned}$$

4. Position Vectors of a General Helix with Respect to Standart Frame

Theorem 4.1. *Let $\alpha(x)$ be a general helix with $k_1(x) \neq 0$, then its tangent vector makes a constant angle with a fixed straight line in space. So, $\alpha(x)$ is determined in the natural representation form in the following*

$$\begin{aligned}
\alpha(x) = \int \Bigg[\int \Bigg(& -k_1(x) \cos\left(1 + \frac{{k_2}^2(x)}{k_1(x)}\right) \int k_1(x)\,dx, \\
& -k_1(x) \sin\left(1 + \frac{{k_2}^2(x)}{k_1(x)}\right) \int k_1(x)\,dx, 0 \Bigg) dx \Bigg] dx
\end{aligned} \tag{4.1}$$

Proof. Let $\alpha(x)$ be a general helix. Using the third equation of (2.3) we obtain

$$N_1(x) = \left(\frac{1}{k_1(x)}\right) B'(x) - \left(\frac{k_2(x)}{k_1(x)}\right) N_2(x) \tag{4.2}$$

Moreover the first equation of (2.3) we have

$$N_1'(x) + k_1(x) B'(x) = 0 \tag{4.3}$$

If we can change the variable x by the variable $\theta = \int k_1(x)\,dx$. So, all functions of x will turn into the functions of θ. Then differentiating the equation (4.2) and using the equation (4.3) we get the following differential equation

$$\ddot{B} + \left(1 + \frac{{k_2}^2}{k_1}\right) B = 0 \tag{4.4}$$

where the dot denote the derivative according to θ. If N_1 is taken instead of tangent vector and α be a general helix, then N_1 makes a constant angle ϕ with the constant vector called the axis of helix. So, without loss of generality, we take the axis to be parallel to e_3. Then,

$$N_1 = (N_1)_1(\theta)\, e_1 + (N_1)_2(\theta)\, e_2 + (N_1)_3(\theta)\, e_3$$

and

$$(N_1)_3(\theta) = \langle N_1, e_3 \rangle = cos(\phi) = n. \tag{4.5}$$

Also we have

$$B = B_1(\theta)\, e_1 + B_2(\theta)\, e_2 + B_3(\theta)\, e_3 \tag{4.6}$$

and differentiating the equation (4.5) we get

$$B_3(\theta) = \langle B, e_3 \rangle = 0$$

Because of B is a unit vector, then the condition that must be satisfied is as follows

$$B_1^2(\theta) + B_2^2(\theta) = 1$$

The general solution of this equation can be given by

$$B_1 = \cos\left[t\left(\theta\right)\right], \qquad B_2 = \sin\left[t\left(\theta\right)\right],$$

where t is an arbitrary function of θ. So, substiuting the components of B in the equation (4.4) we obtain

$$-\ddot{t}\sin\left(t\right) + \left[1 + \left(\frac{{k_2}^2}{k_1}\right) - \dot{t}\right]\cos\left(t\right) = 0 \tag{4.7}$$

$$\ddot{t}\cos\left(t\right) + \left[1 + \left(\frac{{k_2}^2}{k_1}\right) - \dot{t}\right]\sin\left(t\right) = 0. \tag{4.8}$$

The above equations can be reduced in the following

$$\ddot{t} = 0 \tag{4.9}$$

$$\left[1 + \left(\frac{{k_2}^2}{k_1}\right)\right] - \dot{t} = 0 \tag{4.10}$$

The general solution of equation (4.9) is

$$t\left(\theta\right) = c_1\theta + c_2 \tag{4.11}$$

where c_1 and c_2 are constants. Using the equation (4.11) in (4.10) we obtain

$$c_1 = \left[1 + \left(\frac{{k_2}^2}{k_1}\right)\right]$$

So, we can write

$$t\left(\theta\right) = \left[1 + \left(\frac{{k_2}^2}{k_1}\right)\right]\theta + c_2$$

The constants of c_2 are disappear when making a change such that $t \to t + c_2$. Therefore the unit normal vector B is find as follows

$$B = \left(\cos\left[1 + \left(\frac{{k_2}^2}{k_1}\right)\right]\int k_1\left(x\right)dx, \sin\left[1 + \left(\frac{{k_2}^2}{k_1}\right)\right]\int k_1\left(x\right)dx, 0\right) \tag{4.12}$$

Considering the equation (2.3) and (4.12) we obtain the equation (4.1) which complete the proof. □

Lemma 4.1. *Let $\alpha\left(x\right)$ be a circular helix, So, $\alpha\left(x\right)$ is determined in the natural representation form in the following*

$$\alpha\left(x\right) = \int\left[-k_1\int\left(\cos\left({k_1}^2 + {k_2}^2\right)x, \sin\left({k_1}^2 + {k_2}^2\right)x, 0\right)dx\right]dx \tag{4.13}$$

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