SOME CONDITIONS ON STARLIKE AND CLOSE TO CONVEX FUNCTIONS

by

Ismet YILDIZ[a,*], Hasan SAHIN[a], and Oya MERT[b]

[a] Department of Mathematics, Faculty of Arts and Sciences, Duzce University, Duzce, Turkey
[b] Department of Mathematics, Faculty of Arts and Sciences, Tekirdag Namik Kemal University, Tekirdag, Turkey

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Many mathematical concepts are explained when viewed through complex function theory. We are here basically concerned with the form $f(z)=a_0+a_1z+a_2z^2+\ldots$ $f(z)\in A$, $f(z)=z+\Sigma_{n=2}^{\infty}a_nz^n$ *will be an analytic function in the open unit disc* $U=\{z:|z|<1, z\in\mathbb{C}\}$ *normalized by* $f(0) = 0$, $f'(0) = 1$. *In this work, starlike functions and close-to-convex functions with order 1/4 have been studied according to the exact analytic requirements.*

Key words: *analytic function, univalent function, close to convex function, starlike function*

Introduction

Let A be the class of analytic functions $f(z)$ in the unit disc $U=\{z:|z|<1, z\in\mathbb{C}\}$ normalized by:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \text{ for } f(0)=0,\ f'(0)=1$$

where S denotes the class of $f(z)$ functions in A which $f(z)$ is a univalent function. These $f(z)\in A$ functions lie in U as starlike of order $\alpha(0\le\alpha<1)$, such that:

$$\text{Re}\left[\frac{zf'(z)}{f(z)}\right]>\alpha,\quad f(z)\in A \text{ for all } z\in U=\{z:|z|<1, z\in\mathbb{C}\}$$

In other words $f(z)\in S^*(\alpha)$. That is $f(z)\in S^*(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$. If there is a convex function $g(z)$ that provides the following function, then the $f(z)$ is called close-to-convex. Let K^* be the class of close-to-convex:

$$\text{Re}\left[\frac{zf'(z)}{g(z)}\right]>\alpha,\quad z\in U=\{z:|z|<1, z\in\mathbb{C}\}$$

According to the definitions for the class starlike functions $S^*(\alpha)$ and complex functions $K(\alpha)$ which these functions are of α degree. We know that $f(z)\in K(\alpha)$ if and only if

* Corresponding author, e-mail: ismetyildiz@duzce.edu.tr

$zf'(z) \in S^*(\alpha)$ [1-5]. For the starlike function $f(z)$ with a degree $\alpha(0 \le \alpha < 1)$, we can give the following function as an example:

$$f(z) = \frac{z}{1-z^2} = z\sum_{n=2}^{\infty} nz^{(2n-1)} \in S^*$$

where $f(U)$ is starlike region by origin and for the convex function $f(z)$ with $\alpha(0 \le \alpha < 1)$, we can give the following function as an example:

$$f(z) = \frac{1}{2}, \quad \ln\left(\frac{1+z}{1-z}\right) = z + \sum_{n\ge 2}^{\infty} \frac{1}{2n-1} z^{2n-1} \in K$$

which *K* is set of convex functions where $f(U)$ is convex region in complex plane.

Definition 1. Let $f \in f_\alpha$ with the relevant domain *D*. We denote by $P_\alpha(f)$ and our functions $p(z) = \alpha + p_1 z + p_2 z^2 + \ldots$ that are regular in domain *D* and satisfy:

$$[p(z), zp'(z)] \in D \text{ and } \mathrm{Re} f[p(z), zp'(z)] > 0$$

when $z \in D$. Here the class $P_\alpha(f)$ is not empty since for any $f \in f_\alpha$ it is true that $p(z) = \alpha + p_1 z \in P_\alpha(f)$ for $|p_1|$ sufficiently small (depends on *f*) [6].

Theorem 1. Let $p(z) \in A$ and suppose that there exists a point $z_0 \in D$ such that:

$$\mathrm{Re}p(z) > 0 \quad \text{for} \quad |z| < z_0,$$

$$\mathrm{Re}p(z_0) = 0 \quad \text{and} \quad p(z_0) \ne 0$$

Then we have:

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where *k* is real and $|k| \ge 1$ [7, 8].

Lemma 1. Let $h(z) = 1 + \Sigma_{n=2}^{\infty} c_n z^n$ be analytic in the unit disc *U* and suppose that there exists a point $z_0 \in U$ such that:

$$\mathrm{Re}h(z) > 0 \quad \text{and} \quad \mathrm{Re}h(z_0) = 0$$

Then we have [9]:

$$z_0 h'(z_0) \le -\frac{1}{2}\left[1 + |h(z_0)|^2\right] \quad \text{for} \quad |z| < |z_0|$$

Theorem 2 (Main *Theorem 1*). Let $f(z)$ and suppose that there exists a starlike function *g*(*z*) such that:

$$\mathrm{Re}\left\{\frac{z.f'(z)}{g(z)}\left[1 + \frac{z.f''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right]\right\} > -\frac{1}{8}\left(1 + \left|\frac{zf'(z)}{f(z)}\right|^2\right)$$

then $f(z)$ is a close-to-convex function of degree 1/4 *i.e* $f(z) \in K^*(1/4)$.

Proof. Let put:

$$h(z) = 4\left[\frac{zf'(z)}{g(z)} - \frac{3}{4}\right] \text{ for } h(0) = 1$$

Then $h(z)$ is analytic in $|z|<1$ which satisfies the condition. Now by using $h(z)=4\{[zf'(z)]/[g(z)]\}-3/4$:

$$h'(z)=4\left\{\frac{[f'(z)+zf''(z)].g(z)-g'(z)zf'(z)}{[g(z)]^2}\right\}$$

$$zh'(z)=4\left[\frac{zf'(z)}{g(z)}+\frac{zf''(z)}{f'(z)}\frac{zf'(z)}{g(z)}-\frac{zf'(z)}{g(z)}\frac{zg'(z)}{g(z)}\right]=4\left\{\frac{zf'(z)}{g(z)}\left[1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right]\right\}$$

$$\frac{1}{4}zh'(z)=\left\{\frac{zf'(z)}{g(z)}\left[1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right]\right\}$$

from $h(z)$ is analytic in U and $h(0) = 1$ suppose that there exists a complex number $z_0 \in U$ which satisfies the conditions of lemma and from here:

$$\left\{\frac{zf'(z)}{g(z)}\left[1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right]\right\}=\frac{1}{4}zh'(z)$$

On the other hand, since the function $h(z)$ and the point $z_0 \in |z|<1$ satisfy all conditions of *Lemma 1*, then we obtain:

$$\mathrm{Re}\left(\frac{z_0 f'(z_0)}{g(z_0)}\left\{\left(\left[1+\frac{z_0 f''(z_0)}{f'(z_0)}-\frac{z_0 g'(z_0)}{g(z_0)}\right]\le -\frac{1}{8}\left(1+\left|h(z_0\right|^2)\right)\right)\right\}\right)=-\frac{1}{8}\left[1+\left|\frac{z_0 f'(z_0)}{g(z_0)}\right|^2\right]$$

Therefore proof of theorem (Main *Theorem 1*) is completed.

Theorem 3. Let $f(z)\in A,$ and suppose that there exist a starlike function $g(z)$such that:

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\left[1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right]\right\}>-\frac{1}{2}\left(1+\left|\frac{zf'(z)}{g(z)}\right|^2\right)\quad \text{for}\quad z_0\in|z|<1,$$

then $f(z)$ is the close-to-convex, so $f(z)\in K^*$.

Proof. If $h(z)=[zf'(z)]/[g(z)]$ and $h(z)$ is analytic in U. By using $h(z)=[zf'(z)]/[g(z)]$, we have

$$\frac{z_0 f'(z_0)}{g(z_0)}\left[1+\frac{z_0 f''(z_0)}{f'(z_0)}-\frac{zg'(z_0)}{g(z_0)}\right]=z_0h'(z_0)$$

Therefore, we obtain [9]:

$$\mathrm{Re}\left\{\frac{z_0 f'(z_0)}{g(z_0)}\left[1+\frac{z_0 . f''(z_0)}{f'(z_0)}-\frac{zg'(z_0)}{g(z_0)}\right]\right\}=$$

$$=z_0h'(z_0)\le -\frac{1}{2}\left(1+\left|h(z_0)^2\right|\right)=-\frac{1}{2}\left(1+\left|\frac{z_0 f'(z_0)}{g(z_0)}\right|^2\right)$$

Lemma 2. Let $h(z)=1+\Sigma_{n=2}^{\infty}c_n z^n$ be analytic in $|z|<1$ and (α which is $0<\alpha\leq 1/2$) be a positive real number. Then suppose that there exists a point $z_0\in|z|<1$ such that [10]:

$$\text{Re}h(z)>\alpha \quad \text{and} \quad \text{Re}h(z_0)=\alpha \quad \text{and} \quad h(z_0)\neq\alpha \quad \text{for} \quad |z|<|z_0|$$

$$\frac{z_0 h'(z_0)}{h(z_0)}\leq-\frac{\alpha}{2(1-\alpha)}$$

Lemma 3. Let $v(z)$ be a nonconstant analytic function in $|z|<1$ with $v(0)=0$. If $|v(z)|$ attaints its maximum value on the $|z|=r<1$ at z_0. Then [1]:

$$z_0 w'(z_0)=kw(z) \quad \text{where} \quad k\geq 1 \text{ is a real number.}$$

Theorem 4. If $f(z)\in A$ satisfies the following inequality:

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\left[1+\alpha\frac{zf''(z)}{f'(z)}\right]\right\}>-\frac{\alpha^2}{4}(1-\alpha), \quad 0\leq\alpha<2$$

then $f(z)\in S^*(1/2)$ [11, 12].

Theorem 5. If $f(z)\in A$ satisfies the following inequality [1, 11]:

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\left[1+\frac{zf''(z)}{f'(z)}\right]\right\}>0\text{, then } f(z)\in S^*\left(\frac{1}{2}\right)$$

Theorem 6. Let $\alpha(0<\alpha\leq 1/3)$ is a positive real number and $f(z)\in A$. If:

$$\text{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>\text{Re}\left[\frac{zf'(z)}{f(z)}\right]-\frac{1}{6}$$

then we have $f(z)\in S^*(1/4)$.

Proof. If $h(z)=[zf'(z)]/[f(z)]$. Then $h(z)$ is analytic in $|z|<1$ and $h(0)$ = 1. Suppose that there exists a complex number $z_0\in|z|<1$ which satisfies the conditions:

$$\text{Re}h(z)>\frac{1}{4} \quad \text{and} \quad \text{Re}h(z_0)=\frac{1}{4} \quad \text{and} \quad h(z_0)\neq\frac{1}{4} \quad \text{for}|z|<|z_0|$$

Really, now using $h(z)=[zf'(z)]/[f(z)]$, it follows that:

$$h'(z)=\frac{[f'(z)+zf'(z)]f(z)-z[f'(z)]^2}{[f(z)]^2}$$

$$zh'(z)=\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\frac{zf'(z)}{f(z)}-\frac{zf'(z)}{f(z)}\frac{zf'(z)}{f(z)} \tag{1}$$

$$\frac{zh'(z)}{h(z)}=1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)} \text{ for } h(0)=\frac{zf'(0)}{f(0)}=1$$

Since the function $h(z)$ and $z_0\in|z|<1$ satisfy all conditions of *Lemma 1*, therefore in view of $[zh'(z)]/[h(z)]$ and (1), it gives:

$$\text{Re}\left[1+\frac{z_0 f''(z_0)}{f'(z_0)}\right]=\text{Re}\left[\frac{z_0 h'(z_0)}{h(z_0)}+h(z_0\right]$$

This is a contradiction and therefore proof of *Theorem 6* is completed.

Theorem 7. Let $h(z)=1+\Sigma_{n=2}^{\infty}c_n z^n$ be analytic in $U=\{z:|z|<1\}$ and suppose that there exists $z_0 \in U$ such that $\text{Re}[h(z)]>0$ for $|z|<|z_0|$, $\text{Re}[h(z_0)]=0$. Then:

$$z_0 h'(z_0) \le -\frac{1}{4}\left[1+|h(z_0)|^2\right]$$

Proof. Let's define $p(z)=2\{[1-h(z)]/[1+h(z)]\}$ function which satisfies the following conditions in its $|z|<|z_0|$ region $p(0)=0$, $|p(z)|<1$ and $|p(z_0)|=1$:

$$p'(z)=2\frac{-h'(z)[1+h(z)-h'(z)][1-h(z)]}{[1+h(z)]^2}=\frac{-4h'(z)}{[1+h(z)]^2}$$

from $\text{Re}h(z)>0$ and $\text{Re}h(z_0)=0$ for:

$$\frac{zp'(z)}{p(z)}=\frac{-4zh'(z)}{[1-h(z)][1+h(z)]}\ge 1$$

or

$$\frac{z_0 p'(z_0)}{p(z_0)}=\frac{-4z_0 h'(z_0)}{[1-h(z_0)][1+h(z_0)]}\ge 1$$

for $|z|<|z_0|$. Therefore, we have $z_0 h'(z_0)\le -1/4[1+|h(z_0)|^2]$.

Theorem 8. Let's assume that the function $f(z)\in A$ satisfies the conditions $f(z)f'(z)\ne 0$ and:

$$\text{Re}\left\{\frac{z.f'(z)}{f(z)}\left[1+\frac{z.f''(z)}{f'(z)}\right]\right\}>-\frac{1}{4}\left|\frac{z.f'(z)}{f(z)}\right|^2 \quad \text{for} \quad 0<|z|<1$$

then $f(z)\in S^*(1/4)$.

Proof. Let's define the function $h(z)=[2zf'(z)]/[f(z)]-1$ which holds $h(0)=1$. Using this value of $h(z)$, we can consider the following equality:

$$\text{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\left[1+\frac{z_0 f''(z_0)}{f'(z_0)}\right]\right\}=\text{Re}\left\{\frac{1}{4}z_0 h'(z_0)+\frac{1}{8}[1+h(z_0)^2]\right\}$$

where $z_0 \in U$ is a complex number which satisfies $\text{Re}h(z)>0$ for $|z|<|z_0|$ and $\text{Re}h(z_0)=0$. By using relations $\text{Re}h(z_0)=0$, $z_o h'(z_o)\le -(1/4)[1+|h(z_0)|^2]$, the following inequality can be written:

$$\text{Re}\left\{\frac{z_0 f'(z_0)}{f(z_0)}\left[1+\frac{z_0 . f''(z_0)}{f'(z_0)}\right]\right\}\le -\frac{1}{8}[1+|h(z_0)|^2]-\frac{1}{8}|h(z_0)|^2+\frac{1}{8}\le$$

$$\le -\frac{1}{4}|h(z_0)|^2 \le -\frac{1}{4}\left|\frac{z_0 f'(z_0)}{f(z_0)}\right|$$

therefore, we have $\text{Re}h(z)>0$ or $\text{Re}[zf'(z)]/[f(z)]>(1/4)$, so $f(z)\in S^*(1/4)$ [11].

Theorem 9 (Main *Theorem 2*). If $f(z) \in A$ is a function which satisfies the following conditions $zf'(z) \neq 0$ and:

$$\operatorname{Re}\left\{\frac{z.f'(z)}{f(z)}\left[1+\alpha\frac{z.f''(z)}{f'(z)}\right]\right\} > -\frac{\alpha}{8}(3-\alpha)(2-\alpha)(1-\alpha) \text{ in } 0<|z|<1 \text{ for } 0<\alpha<3$$

then $f(z)$ is $1/4$ order starlike function. That is $f(z) \in S^*(1/4)$.

Proof: Let's define $[zf'(z)]/[f(z)] = [1-h(z)](\alpha/4)+h(z)$ for $h(0)=1$, then the following equality can be written such that $\operatorname{Re}h(z)>0$ and $\operatorname{Re}h(z_0)=0$ for $|z|<|z_0|$. Then from here:

$$\frac{d}{dz}\frac{zf'(z)}{f(z)} = \frac{f'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\frac{f'(z)}{f(z)} - z\left(\frac{f'(z)}{f(z)}\right)^2 = \left(1-\frac{\alpha}{4}\right)h'(z)$$

$$\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\frac{zf'(z)}{f(z)} - \left[\frac{zf'(z)}{f(z)}\right]^2 = \left(1-\frac{\alpha}{4}\right)zh'(z)$$

$$\frac{zf'(z)}{f(z)}\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right] = \left(1-\frac{\alpha}{4}\right)zh'(z)$$

Suppose that there exists a complex point $z_0 \in |z|<1$ such that it satisfies the conditions $\operatorname{Re}h(z)>0$ and $\operatorname{Re}h(z_0)=0$ and *Lemma 1, Lemma 2,* and *Lemma 3*. Then we have:

$$\operatorname{Re}\left\{\frac{z_0.f'(z_0)}{f(z_0)}\left[1+\alpha\frac{z_0.f''(z_0)}{f'(z_0)}\right]\right\} =$$

$$= \operatorname{Re}\left\{\alpha\left(1-\frac{\alpha}{4}\right)z_0h'(z_0) + \alpha\left(1-\frac{\alpha}{4}\right)^2[h(z)]^2 + \left(1-\frac{\alpha}{4}\right)(\alpha^2+1-\alpha)h(z_0) + \frac{\alpha^3}{8} + (1-\alpha)\frac{\alpha}{4}\right\}$$

If the relations $\operatorname{Re}h'(z_0)=0$ and $z_0\operatorname{Re}h'(z_0) \le -(1/4)[1+|h(z_0)|^2]$ are used in the previous equation, then:

$$\operatorname{Re}\left\{\frac{z_0.f'(z)}{f(z_0)}\left[1+\alpha\frac{z_0f''(z_0)}{f'(z_0)}\right]\right\} = \operatorname{Re}\left[\frac{z_0f'(z)}{f(z_0)}\right]\operatorname{Re}\left[1+\alpha\frac{z_0f''(z_0)}{f'(z_0)}\right] \le$$

$$\le -\frac{\alpha}{4}\left(1-\frac{\alpha}{4}\right)(1+|h(z_0)|^2) - \alpha(1-\frac{\alpha}{4})^2|h(z_0)|^2 + \frac{\alpha^3}{8} + \frac{\alpha}{4}(1-\alpha) \le$$

$$\le -\frac{\alpha}{4}\left(1-\frac{\alpha}{4}\right) + \frac{\alpha^3}{8} + \frac{\alpha}{4}(1-\alpha) \le -\frac{\alpha^3}{8}(3-\alpha)(2-\alpha)(1-\alpha)$$

$\operatorname{Re}[z.f'(z)]/[f(z)] > (\alpha/4)$ is also obtained, which is $f(z) \in S^*(1/4)$.

Conclusion

Theorem 2 is improved from *Theorem 9* because obtained results are:

$$0 > -\frac{\alpha^2}{8}(1-\alpha) > -\frac{\alpha}{8}(3-\alpha)(2-\alpha)-(1-\alpha) \quad \text{when} \quad 0<\alpha<1$$

$$0 > -\frac{\alpha^2}{8}(1-\alpha) > -\frac{\alpha}{8}(3-\alpha)(2-\alpha)-(1-\alpha) \quad \text{when} \quad 1\le\alpha<2$$

$$0 > -\frac{\alpha^2}{8}(1-\alpha) > -\frac{\alpha}{8}(3-\alpha)(2-\alpha)-(1-\alpha) \quad \text{when} \quad 2\le\alpha<3$$

References

[1] Nunokawa, M., *et al.,* Sufficient Conditions for Starlikeness, *Journal of Classical Analysis, 1* (2012), 4, pp 85-90

[2] Nunokawa, M., *et al.,* Some Properties Concerning Close-to-Convexity of Certain Analytic Functions, *Journal of Inequalities and Applications, 245* (2012), Oct., pp. 1-10

[3] Nunokaw, M., *et al.,* Sufficient Conditions for Starlikeness, *Chinese Journal of Mathematics, 24* (1996), 3, pp. 265-270

[4] Yildiz, I., *et al.,* An Analytical Investigation on Starlikeness and Convexity Properties for Hypergeometric Functions, *Turkish Journal of Mathematics, 44* (2020), 4, pp. 1453-1465

[5] Xu, Q-H., *et al.,* Coefficient Estimates for Certain Subclasses of Analytic Functions of Complex Order, *Taiwanese Journal of Mathematics, 15* (2011), 5, pp. 2377-2386

[6] Lewandowski, Z., *et al.,* Generating Functions for Some Classes of Univalent Functions, *Proceedings of the American Mathematical Society, 56* (1976), 1, pp. 111-117

[7] Nunokawa, M., On Properties of Non-Caratheodory Functions, *Proceedings of the Japan Academy*, *68* (1992), 6, pp. 152- 153

[8] Sudharsan, T. V., *et al.,* Third Hankel Determinant for a Subclass of Analytic Univalent Functions, *Malaya J. Mat.,2* (2014), 4, pp. 438- 444

[9] Liu, H., Sufficient Conditions for Certain Analytic Functions, *Scholars Journal of Engineering and Technology (SJET), 2* (2014), 2B, pp. 243-246

[10] Guo, L., *et al.,* On the Starlikeness for Certain Analytic Functions, *International Journal of Mathematical Research, 1* (2012), 2, pp. 21-25

[11] Li, J., *et al.,* Sufficient Conditions for Starlikeness, *Indian Journal of Pure and Applied Mathematics, 33* (2002), 9, pp. 1385-1390

[12] Sokol, J., *et al.,* On Some Sufficient Conditions for Univalence and Starlikeness, *Journal of Inequalities and Applications, 282* (2012), 1, pp. 282-291

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